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Properties of Monotonic Effects on Directed Acyclic Graphs

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Abstract

Various relationships are shown hold between monotonic effects and weak monotonic effects and the monotonicity of certain conditional expectations. Counterexamples are provided to show that the results do not hold under less restrictive conditions. Monotonic effects are furthermore used to relate signed edges on a causal directed acyclic graph to qualitative effect modification. The theory is applied to an example concerning the direct effect of smoking on cardiovascular disease controlling for hypercholesterolemia. Monotonicity assumptions are used to construct a test for whether there is a variable that confounds the relationship between the mediator, hypercholesterolemia, and the outcome, cardiovascular disease.

Keywords: Bayesian networks, conditional expectation, covariance, directed acyclic graphs, effect modification, monotonicity

1. Introduction

Several papers have considered various monotonicity relationships on Bayesian networks or directed acyclic graphs. Wellman (1990) introduced the notion of qualitative causal influence and derived various resulting concerning the propagation of qualitative influences, the preservation of monotonicity under edge reversal, the necessity of first order stochastic dominance for propagating influences and the propagation of sub-additive and super-additive relationships on probabilistic networks. Druzdzel and Henrion (1993) developed a polynomial time algorithm for reasoning in qualitative probabilistic network, based on local sign propagation. More recently, van der Gaag et al. (2004) showed that identifying whether a network exhibits various monotonicity properties is coNP^{PP}- complete. VanderWeele and Robins (2009) introduced the concept of a monotonic effect which is closely related to Wellman's qualitative influence and considered the relationship between monotonicity properties and causal effects, covariance, bias and confounding. In this paper we develop a number of probabilistic properties concerning monotonic effects and weak monotonic effects. Some of these properties give rise to certain inequality constraints that could be used to test for the presence of hidden or unmeasured confounding variables. These inequality constraints which arise from monotonicity relationships provide constraints beyond those already available in the literature (Kang and Tian, 2006). The paper is organized as follows. In Section 2 we describe

the notation we will use in this paper and review the definitions concerning directed acyclic graphs. In Section 3 we present a motivating example for the theory that will be developed. In Section 4, we define the concepts of a monotonic effect and a weak monotonic effect in the directed acyclic graph causal framework, the latter essentially being equivalent to Wellman's (1990) qualitative influence. In Section 5, we give a number of results relating weak monotonic effects to the monotonicity in the conditioning argument of certain conditional expectations; we also return to the motivating example and show how the theory developed can be applied to this example. Finally, in Section 6, we give a number of results that relate weak monotonic effects to the existence of qualitative effect modifiers. Section 7 closes with some concluding remarks.

2. Notation and Directed Acyclic Graphs

Following Pearl (1995), a causal directed acyclic graph is a set of nodes $(X_1,...,X_n)$ and directed edges amongst nodes such that the graph has no cycles and such that for each node X_i on the graph the corresponding variable is given by its non-parametric structural equation $X_i = f_i(pa_i, \varepsilon_i)$ where pa_i are the parents of X_i on the graph and the ε_i are mutually independent. We will use Ω to denote the sample space for ε and ω to denote a particular point in the sample space. These non-parametric structural equations can be seen as a generalization of the path analysis and linear structural equation models (Pearl, 1995, 2000) developed by Wright (1921) in the genetics literature and Haavelmo (1943) in the econometrics literature. Directed acyclic graphs can be interpreted as representing causal relationships. The non-parametric structural equations encode counterfactual relationships amongst the variables represented on the graph. The equations themselves represent one-step ahead counterfactuals with other counterfactuals given by recursive substitution. The requirement that the ε_i be mutually independent is essentially a requirement that there is no variable absent from the graph which, if included on the graph, would be a parent of two or more variables (Pearl, 1995, 2000). Further discussion of the causal interpretation of directed acyclic graphs can be found elsewhere (Pearl, 1995, 2000; Spirtes et al., 2000; Dawid, 2002; Robins, 2003).

A path is a sequence of nodes connected by edges regardless of arrowhead direction; a directed path is a path which follows the edges in the direction indicated by the graph's arrows. A node C is said to be a common cause of A and Y if there exists a directed path from C to Y not through A and a directed path from C to A not through Y. We will say that $V_1,...,V_n$ constitutes an ordered list if $i < j$ implies that V_i is not a descendent of V_j. A collider is a particular node on a path such that both the preceding and subsequent nodes on the path have directed edges going into that node, that is, both the edge to and the edge from that node have arrowheads into the node. A path between A and B is said to be blocked given some set of variables Z if either there is a variable in Z on the path that is not a collider or if there is a collider on the path such that neither the collider itself nor any of its descendants are in Z. If all paths between A and B are blocked given Z then A and B are said to be d-separated given Z. It has been shown that if A and B are d-separated given Z then A and B are conditionally independent given Z (Verma and Pearl, 1988; Geiger et al., 1990; Lauritzen et al., 1990). We will use the notation $A \coprod B|Z$ to denote that A is conditionally independent of B given Z; we will use the notation $(A \coprod B|Z)_G$ to denote that A and B are d-separated given Z on graph G. The directed acyclic graph causal framework has proven to be particularly useful in determining whether conditioning on a given set of variables, or none at all, is sufficient to control for confounding. The most important result in this regard is the back-door path criterion (Pearl, 1995). A back-door path from some node A to another node Y is a path which begins with a directed edge into A. Pearl

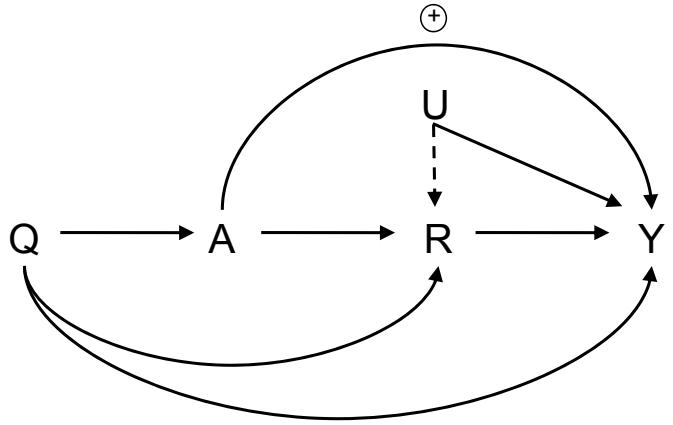

Figure 1: Motivating example concerning the estimation of controlled direct effects.

(1995) showed that for intervention variable A and outcome Y, if a set of variables Z is such that no variable in Z is a descendent of A and such that Z blocks all back-door paths from A to Y then conditioning on Z suffices to control for confounding for the estimation of the causal effect of A on Y. The counterfactual value of Y intervening to set $A = a$ we denote by $Y_{A=a}$.

3. Motivating Example

To motivate the theory we develop in this paper consider the following example.

Example 1. Suppose that Figure 1 represents a causal directed acyclic graph. Let A denote smoking; let R hypercholesterolemia; and let Y denote cardiovascular disease. High cholesterol can lead to the narrowing of the arteries resulting in cardiovascular disease; smoking can lead to blood clots through platelet aggregation resulting in cardiovascular disease. Let Q denote some variable that confounds the relationships between smoking and cardiovascular disease and between hypercholesterolemia and cardiovascular disease (e.g., stress). Let U be some unmeasured variable which might confound the relationship between hypercholesterolemia and cardiovascular disease. The researcher is unsure whether the variable U is a cause of R and we therefore represent the edge from U to R as a dashed line. The results of Pearl (2001) imply that it is possible to estimate controlled direct effects of the form $Y_{A=a_1,R=r} - Y_{A=a_0,R=r}$ (i.e., the direct effect of smoking on cardiovascular disease controlling for hypercholesterolemia) on the graph in Figure 1 if that U is not a cause of R. Suppose that although the researcher is unsure about the presence an edge from U to R, it is known that the relationship between A and Y is monotonic in the sense that $P(Y > y|A = a, R = r, Q = q, U = u)$ is non-decreasing in a for all y, r, q and u. In Section 5, we will present theory that will allow us to derive a statistical test for the null hypothesis that there is no unmeasured variable U confounding the relationship between R and Y.

4. On the Definition of a Monotonic Effect

The definition of a monotonic effect is given in terms of a directed acyclic graph's nonparametric structural equations.

Definition 1. *The non-parametric structural equation for some node Y on a causal directed acyclic graph with parent A can be expressed as* $Y = f(\widetilde{pa}_Y, A, \varepsilon_Y)$ *where* $\widetilde{pa}_Y$ *are the parents of Y other than A; A is said to have a positive monotonic effect on Y if for all* $\widetilde{pa}_Y$ *and* ε_Y, $f(\widetilde{pa}_Y, A_1, \varepsilon_Y) \geq f(\widetilde{pa}_Y, A_2, \varepsilon_Y)$ *whenever* $A_1 \geq A_2$. *Similarly A is said to have a negative monotonic effect on Y if for all* $\widetilde{pa}_Y$ *and* ε_Y, $f(\widetilde{pa}_Y, A_1, \varepsilon_Y) \leq f(\widetilde{pa}_Y, A_2, \varepsilon_Y)$ *whenever* $A_1 \geq A_2$.

As we have defined it above, a causal direct acyclic graph corresponds to a set of non-parametric structural equations and as such the definition of a monotonic effect given above is relative to a particular set of non-parametric structural equations. The presence of a monotonic effect is closely related to the monotonicity of counterfactual variables as is made clear by the following proposition. All proofs of all propositions and lemmas are given in Appendix A.

Proposition 1. *The variable A has a positive monotonic effect on Y if and only if for all* ω *and all values of* $\widetilde{pa}_Y$, $Y_{a_1,\widetilde{pa}_Y}(\omega) \geq Y_{a_0,\widetilde{pa}_Y}(\omega)$ *whenever* $a_1 \geq a_0$.

We note that several sets of non-parametric structural equations may yield identical distributions of $X = (X_1, ..., X_n)$ and $\{X_{V=v}\}_{V \subseteq X, v \in supp(V)}$ (Pearl, 2000). In the context of characterizations of causal directed acyclic graphs that make reference to counterfactuals but not to non-parametric structural equations (Robins, 2003), a positive monotonic effect could instead be defined to be present if for all $\widetilde{pa}_Y$ and $a_1 \geq a_0$, $P(Y_{a_1,\widetilde{pa}_Y} \geq Y_{a_0,\widetilde{pa}_Y}) = 1$. If this latter condition holds with respect to one set of non-parametric structural equations it will hold for any set of non-parametric structural equations which yields the same distribution for X and $\{X_{V=v}\}_{V \subseteq X, v \in supp(V)}$. We note that if for $a_1 \geq a_0$ the set $\{\omega : Y_{a_1,\widetilde{pa}_Y}(\omega) < Y_{a_0,\widetilde{pa}_Y}(\omega)\}$ is of measure zero then $Y_{a_1,\widetilde{pa}_Y}$ and $Y_{a_0,\widetilde{pa}_Y}$ could be re-defined on this set so that $Y_{a_1,\widetilde{pa}_Y}(\omega) \geq Y_{a_0,\widetilde{pa}_Y}(\omega)$ for all ω and so that the distributions of X and $\{X_{V=v}\}_{V \subseteq X, v \in supp(V)}$ remain unchanged.

Because for any value ω we observe the outcome only under one particular value of the intervention variable, the presence of a monotonic effect is not identifiable. The results presented in this paper are in fact true under slightly weaker conditions which are identifiable when data on all of the directed acyclic graph's variables are observed. We thus introduce the concept of a weak monotonic effect which is a special case of Wellman's positive qualitative influence (Wellman, 1990). The definition of a weak monotonic effect does not make reference to counterfactuals and thus can be used in characterizations of causal directed acyclic graphs that do not employ the concept of counterfactuals (Spirtes et al., 2000; Dawid, 2002). The stronger notion of a monotonic effect given above is useful in the context of testing for synergistic relationships (VanderWeele and Robins, 2008).

Definition 2. *Suppose that variable A is a parent of some variable Y and let* $\widetilde{pa}_Y$ *denote the parents of Y other than A. We say that A has a weak positive monotonic effect on Y if the survivor function* $S(y|a, \widetilde{pa}_Y) = P(Y > y|A = a, \widetilde{pa}_Y)$ *is such that whenever* $a_1 \geq a_0$ *we have* $S(y|a_1, \widetilde{pa}_Y) \geq S(y|a_0, \widetilde{pa}_Y)$ *for all y and all* $\widetilde{pa}_Y$*; the variable A is said to have a weak negative monotonic effect on Y if whenever* $a_1 \geq a_0$ *we have* $S(y|a_1, \widetilde{pa}_Y) \leq S(y|a_0, \widetilde{pa}_Y)$ *for all y and all* $\widetilde{pa}_Y$.

Proposition 2. *If A has a positive monotonic effect on Y then A has a weak positive monotonic effect on Y.*

We note that for parent A and child Y, the definition of a weak monotonic effect coincides with Wellman's (1990) definition of positive qualitative influence when the "context" for qualitative influence is chosen to be the parents of Y other than A.

A monotonic effect is a relation between two nodes on a directed acyclic graph and as such it is associated with an edge. The definition of the sign of an edge can be given either in terms of monotonic effects or weak monotonic effects. We can define the sign of an edge as the sign of the monotonic effect or weak monotonic effect to which the edge corresponds; this in turn gives rise to a natural definition for the sign of a path.

Definition 3. *An edge on a causal directed acyclic graph from X to Y is said to be of positive sign if X has a positive monotonic effect on Y. An edge from X to Y is said to be of negative sign if X has a negative monotonic effect on Y. If X has neither a positive monotonic effect nor a negative monotonic effect on Y, then the edge from X to Y is said to be without a sign.*

Definition 4. *The sign of a path on a causal directed acyclic graph is the product of the signs of the edges that constitute that path. If one of the edges on a path is without a sign then the sign of the path is said to be undefined.*

We will call a causal directed acyclic graph with signs on those edges which allow them a signed causal directed acyclic graph. The theorems in this paper are given in terms of signed paths so as to be applicable to both monotonic effects and weak monotonic effects. One further definition will be useful in the development of the theory below.

Definition 5. *Two variables X and Y are said to be positively monotonically associated if all directed paths from X to Y or from Y to X are of positive sign and all common causes C_i of X and Y are such that all directed paths from C_i to X are of the same sign as all directed paths from C_i to Y; the variables X and Y are said to be negatively monotonically associated if all directed paths between X and Y are of negative sign and all common causes C_i of X and Y are such that all directed paths from C_i to X are of the opposite sign as all directed paths from C_i to Y.*

It has been shown elsewhere (VanderWeele and Robins, 2009) that if X and Y are positively monotonically associated then $Cov(X,Y) \geq 0$ and if X and Y are negatively monotonically associated then $Cov(X,Y) \leq 0$. We now develop several results concerning the monotonicity in the conditioning argument of certain conditional expectations.

5. Monotonic Effects and Conditional Expectations

Lemma 1 below can be proved by integration by parts and will be used in the proofs of the subsequent propositions. We will assume throughout the remainder of this paper that the random variables under consideration satisfy regularity conditions that allow for the integration by parts required in the proof of Lemma 1. If conditional cumulative distribution functions are continuously differentiable then the regularity conditions will be satisfied; the regularity conditions will also be satisfied if all variables are discrete. Härdle et al. (1998, p72) also gives relatively weak conditions under which such integration by parts is possible. Alternatively, the existence of the Lebesgue-Stieltjes integrals found in the proof of Lemma 1 suffices to allow integration by parts. Note that Lemma 1

will always be applied either to the function $h(y,a,r) = y$ or to conditional survivor functions which will satisfy the relevant regularity conditions; thus the conditions which are required for integration by parts are only regularity conditions on the distribution of the random variables.

Lemma 1. *If $h(y,a,r)$ is non-decreasing in y and in a and $S(y|a,r) = P(Y > y|A = a, R = r)$ is non-decreasing in a for all y then $E[h(Y,A,R)|A = a, R = r]$ is non-decreasing in a.*

Proposition 3 immediately follows from Lemma 1.

Proposition 3. *Suppose that the A–Y edge, if it exists, is positive. Let X denote some set of non-descendents of Y that includes $\widetilde{pa}_Y$, the parents of Y other than A, then $E[Y|X = x, A = a]$ is non-decreasing in a for all values of x.*

Proposition 4 gives the basic result for the monotonicity of conditional expectations. For the conditional expectation of some variable Y to be monotonic in a conditioning argument A, it requires that the conditioning set includes variables that block all backdoor paths from A to Y. In order to prove Proposition 4 we will make use of the following two lemmas.

Lemma 2. *Suppose that A is a non-descendent of Y and let Q denote the set of ancestors of A or Y which are not descendents of A. Let $R = (R_1, ..., R_m)$ denote an ordered list of some set of nodes on directed paths from A to Y such that for each i the backdoor paths from R_i to Y are blocked by $R_1, ..., R_{i-1}, A$, and Q. Let $V_0 = A$ and $V_n = Y$ and let $V_1, ..., V_{n-1}$ be an ordered list of all the nodes which are not in R but which are on directed paths from A to Y such that at least one of the directed paths from each node to Y is not blocked by R. Let $\overline{V}_k = \{V_1, ..., V_k\}$ then $S(v_k|a, \bar{v}_{k-1}, q, r) = S(v_k|pa_{v_k})$.*

Lemma 3. *If under the conditions of Lemma 2 all directed paths from A to Y are positive except possibly through R then $S(y|a,q,r)$ is non-decreasing in a.*

These two lemmas allow us to prove Proposition 4 given below.

Proposition 4. *Suppose that A is a non-descendent of Y and let X denote some set of non-descendents of A that blocks all backdoor paths from A to Y. Let $R = (R_1, ..., R_m)$ denote an ordered list of some set of nodes on directed paths from A to Y such that for each i the backdoor paths from R_i to Y are blocked by $R_1, ..., R_{i-1}, A$ and X. If all directed paths from A to Y are positive except possibly through R then $S(y|a,x,r)$ and $E[y|a,x,r]$ are non-decreasing in a.*

If $R = \varnothing$ the statement of Proposition 4 is considerably simplified and is stated in the following corollary.

Corollary. *Let X denote some set of non-descendents of A that blocks all backdoor paths from A to Y. If all directed paths between A and Y are positive then $S(y|a,x)$ and $E[y|a,x]$ are non-decreasing in a.*

Lemma 3 and Proposition 4 are generalizations of results given by Wellman (1990) and Druzdzel and Henrion (1993). In particular, in Lemma 3 if $R = \varnothing$, then the result follows immediately from repeated application of Theorems 4.2 and 4.3 in Wellman (1990) or more directly from the work of Druzdzel and Henrion (1993, Theorem 4). Lemma 3 generalizes the results of Wellman (1990) and Druzdzel and Henrion (1993) by allowing for conditioning on nodes $R = (R_1, ..., R_m)$ which are on directed paths from A to Y. Proposition 4 further generalizes Lemma 3 by replacing the set Q in

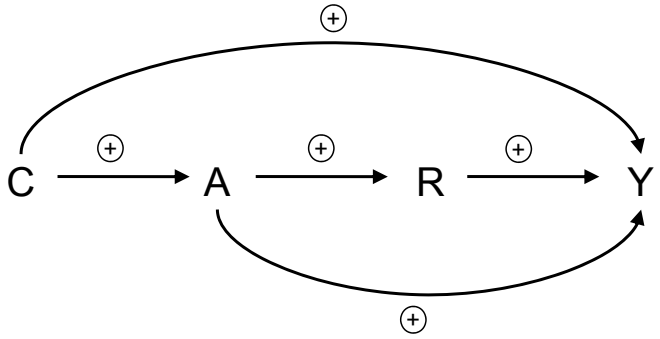

Figure 2: Example illustrating Propositions 4-6.

Lemma 3 which consists of the set of ancestors of A or Y which are not descendents of A with some other set X which consists of some set of non-descendents of A that blocks all backdoor paths from A to Y.

Propositions 5-8 relax the condition that the conditioning set includes variables that block all backdoor paths A to Y and impose certain other conditions; the proofs of each of these propositions make use of Proposition 4.

Proposition 5. *Suppose that A is not a descendent of Y, that A is binary, and that A and Y are positively monotonically associated then $E[Y|A]$ is non-decreasing in A.*

Proposition 6. *Suppose that A is not a descendent of Y, that Y is binary, and that A and Y are positively monotonically associated then $E[A|Y]$ is non-decreasing in Y.*

Propositions 5 and 6 require that the conditioning variable be binary. Counterexamples can be constructed to show that if the conditioning variable is not binary then the conditional expectation may not be non-decreasing in the conditioning argument even if A and Y are positively monotonically associated (see Appendix B, counterexamples 1 and 2).

Propositions 5 and 6 can be combined to give the following corollary which makes no reference to the ordering of A and Y.

Corollary. *Suppose that A is binary and that A and Y are positively monotonically associated then $E[Y|A]$ is non-decreasing in A.*

Example 2. Consider the signed directed acyclic graph given in Figure 2. By Proposition 4, we have that $E[Y|A=a,C=c,R=r]$ and $E[Y|A=a,C=c]$ are non-decreasing in a. If A is binary then by Proposition 5, it is also the case that $E[Y|A=a]$ is non-decreasing in a. If Y is binary, then by Proposition 6, $E[A|Y=y]$ is non-decreasing in y. The monotonicity of $E[Y|A=a,C=c,R=r]$ and $E[Y|A=a,C=c]$ also follow directly from the results of Wellman (1990) and Druzdzel and Henrion (1993); the monotonicity of $E[Y|A=a]$ and $E[A|Y=y]$ do not.

Propositions 7 and 8 consider the monotonicity of conditional expectations while conditioning on variables other than the variable in which monotonicity holds but not conditioning on variables that are sufficient to block all backdoor paths between A and Y. Propositions 7 and 8 generalize

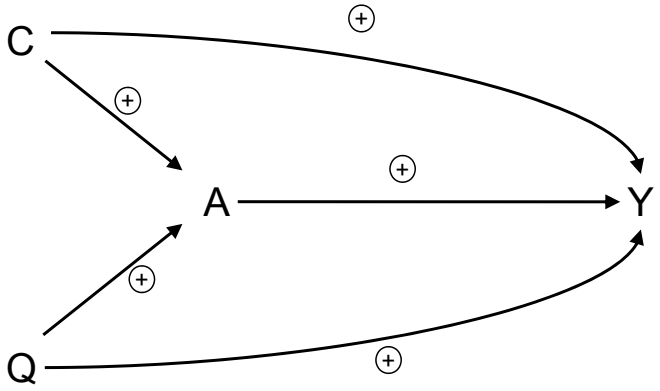

Figure 3: Example illustrating Propositions 7 and 8.

Propositions 5 and 6 respectively.

Proposition 7. *Suppose that A is not a descendent of Y and that A is binary. Let Q be some set of variables that are not descendents of Y nor of A and let C be the common causes of A and Y not in Q. If all directed paths from A to Y are of positive sign and all directed paths from C to A not through Q are of the same sign as all directed paths from C to Y not through $\{Q,A\}$ then $E[Y|A,Q]$ is non-decreasing in A.*

Proposition 8 is similar to Proposition 7 but the conditional expectation $E[A|Y,Q]$ is considered rather than $E[Y|A,Q]$ and Y rather than A is assumed to be binary. The form of the proof differs.

Proposition 8. *Suppose that A is not a descendent of Y and that Y is binary. Let Q be some set of variables that are not descendents of Y nor of A and let C be the common causes of A and Y not in Q. If all directed paths from A to Y are of positive sign and all directed paths from C to A not through Q are of the same sign as all directed paths from C to Y not through $\{Q,A\}$ then $E[A|Y,Q]$ is non-decreasing in Y.*

Propositions 7 and 8 can be combined to give the following corollary which makes no reference to the ordering of A and Y.

Corollary. *Suppose that A is binary. Let Q be some set of variables that are not descendents of Y nor of A and let C be the common causes of A and Y not in Q. If all directed paths from A to Y (or from A to Y) are of positive sign and all directed paths from C to A not through $\{Q,Y\}$ are of the same sign as all directed paths from C to Y not through $\{Q,A\}$ then $E[Y|A,Q]$ is non-decreasing in Y.*

Example 3. Consider the signed directed acyclic graph given in Figure 3. If A is binary, then by Proposition 7, $E[Y|A=a,C=c,Q=q]$, $E[Y|A=a,Q=q]$, $E[Y|A=a,C=c]$ and $E[Y|A=a]$ are all non-decreasing in a. If Y is binary then by Proposition 8, $E[A|Y=y,C=c,Q=q]$, $E[A|Y=y,Q=q]$, $E[A|Y=y,C=c]$ and $E[A|Y=y]$ are all non-decreasing in y. The monotonicity of $E[Y|A=a,C=c,Q=q]$ follows directly from the results of Wellman (1990) and Druzdzel and Henrion (1993); the monotonicity of the other conditional expectations do not.

We now return to Example 1 concerning potential unmeasured confounding in the estimation of controlled direct effects.

Example 1 (Revisited). Consider once again the causal directed acyclic graph given in Figure 1. Suppose that we may assume that A has a weak monotonic effect on Y. Under the null hypothesis that U is not a cause of R (i.e., does not confound the relationship between R and Y) we could conclude by Proposition 4 that $E[Y|A=a,R=r,Q=q]$ is non-decreasing in a for all r and q. Under the alternative hypothesis that U is a cause of R, we could not apply Proposition 4 because of the unblocked backdoor path $R-U-Y$ between R and Y. The monotonicity relationship would thus not necessarily hold. Consequently, if $E[Y|A=a,R=r,Q=q]$ were found not to be monotonic in a then we could reject the null hypothesis that U is not a cause of R. Note that the monotonicity of $E[Y|A=a,R=r,Q=q]$ in a also follows from the results of Wellman (1990) and Druzdzel and Henrion (1993). If, however, there were an edge from U to Q for example, or in more complicated scenarios, the results of Wellman (1990) and Druzdzel and Henrion (1993) would no longer suffice to conclude the monotonicity of $E[Y|A=a,R=r,Q=q]$ in a; one would need to employ Proposition 4.

We now construct a simple statistical test in the case that A, R and Y are all binary (cf. Robins and Greenland, 1992) of the null hypothesis that U is absent from Figure 1. Let n_{ijq} denote the number of individuals in stratum $Q=q$ with $A=i$ and $R=j$ and let let d_{ijq} denote the number of individuals in stratum $Q=q$ with $A=i$ and $R=j$ and $Y=1$. Let p_{ijq} denote the true probability $P(Y=1|A=i,R=j,Q=q)$. From the null hypothesis that U is absent from Figure 1, it follows by Proposition 4 that $p_{1jq}-p_{0jq}\leq 0$ for all j and q. Thus we have $d_{ijq}\sim Bin(n_{ijq},p_{ijq})$ with $\mathbb{E}[\frac{d_{ijq}}{n_{ijq}}]=p_{ijq}$ and $Var(\frac{d_{ijq}}{n_{ijq}})=\frac{p_{ijq}(1-p_{ijq})}{n_{ijq}}$. By the central limit central limit theorem $\frac{(\frac{d_{1jq}}{n_{1jq}}-\frac{d_{0jq}}{n_{0jq}})-(p_{1jq}-p_{0jq})}{\sqrt{\frac{p_{1jq}(1-p_{1jq})}{n_{1jq}}+\frac{p_{0jq}(1-p_{0jq})}{n_{0jq}}}}\dot{\sim}N(0,1)$ and by Slutsky's theorem we have $\frac{(\frac{d_{1jq}}{n_{1jq}}-\frac{d_{0jq}}{n_{0jq}})-(p_{1jq}-p_{0jq})}{\sqrt{\frac{d_{1jq}(n_{1jq}-d_{1jq})}{n_{1jq}^3}+\frac{d_{0jq}(n_{0jq}-d_{0jq})}{n_{0jq}^3}}}\dot{\sim}N(0,1)$. To test the null hypothesis that the edge from U to R is absent from Figure 1 one may thus use the test statistic $\frac{(\frac{d_{1jq}}{n_{1jq}}-\frac{d_{0jq}}{n_{0jq}})}{\sqrt{\frac{d_{1jq}(n_{1jq}-d_{1jq})}{n_{1jq}^3}+\frac{d_{0jq}(n_{0jq}-d_{0jq})}{n_{0jq}^3}}}$ with critical regions of the form: $\{\frac{(\frac{d_{1jq}}{n_{1jq}}-\frac{d_{0jq}}{n_{0jq}})}{\sqrt{\frac{d_{1jq}(n_{1jq}-d_{1jq})}{n_{1jq}^3}+\frac{d_{0jq}(n_{0jq}-d_{0jq})}{n_{0jq}^3}}}>Z_{1-\alpha}\}$ to carry out a one-sided (upper tail) test. The derivation of the power of such a test would require providing explicit structural equations for each of the variables in the model. Similar tests could be constructed for other scenarios. We note that if the test fails to reject the null, one cannot conclude that the arrow from U to R is absent; if the inequality $E[Y|A=a_1,R=r,Q=q]\leq E[Y|A=a_2,R=r,Q=q]$ holds for all $a_1\leq a_2$ this is potentially consistent with both the presence and the absence of an edge from U to R. If, however, the test rejects the null then one can conclude that an edge from U to R must be present, provided the other model assumptions hold. With observational data, the assumption that no unmeasured confounding variable is present can be falsified but it cannot be verified regardless of the approach one takes. It is nevertheless worthwhile testing any empirical implications of the no unmeasured confounding variables assumptions which can be derived, such as those following from Proposition 4.

Tian and Pearl (2002) and Kang and Tian (2007) derived various equality constraints that arise from causal directed acyclic graphs with hidden variables; Kang and Tian (2006) derived various inequality constraints that arise from causal directed acyclic graphs with hidden variables. We note that the inequality constraint $E[Y|A=a_1,R=r,Q=q] \leq E[Y|A=a_2,R=r,Q=q]$ for $a_1 \leq a_2$ does not follow from the results of Tian and Pearl (2002) or Kang and Tian (2006, 2007). The equality and inequality constraints which follow from their work will apply to all causal models consistent with the directed acyclic graph in Figure 1 (without the sign); the inequality constraint $E[Y|A=a_1,R=r,Q=q] \leq E[Y|A=a_2,R=r,Q=q]$ follows only if it can be assumed in Figure 1 that A has a weak positive monotonic effect on Y. More generally, the results in this paper provide a supplementary set of constraints to those of Tian and Pearl (2002) and Kang and Tian (2006, 2007).

6. Effect Modification and Monotonic Effects

If when conditioning on a particular variable, the sign of the effect of another variable on the outcome varies between strata of the conditioning variable, then the conditioning variable is said to be a qualitative effect modifier. The following definition gives the condition for qualitative effect modification more formally.

Definition 6. *A variable Q is said to be an effect modifier for the causal effect of A on Y if Q is not a descendent of A and if there exist two levels of A, a_0 and a_1 say, such that $E[Y_{A=a_1}|Q=q]-E[Y_{A=a_0}|Q=q]$ is not constant in q. Furthermore Q is said to be a qualitative effect modifier if there exist two levels of A, a_0 and a_1, and two levels of Q, q_0 and q_1, such that sign$(E[Y_{A=a_1}|Q=q_1]-E[Y_{A=a_0}|Q=q_1]) \neq$ sign$(E[Y_{A=a_1}|Q=q_0]-E[Y_{A=a_0}|Q=q_0])$.*

Monotonic effects and weak monotonic effects are closely related to the concept of qualitative effect modification. Essentially, the presence of a monotonic effect precludes the possibility of qualitative effect modification. This is stated precisely in Propositions 9 and 10.

Proposition 9. *Suppose that some parent A_1 of Y is such that the A_1-Y edge is of positive sign then there can be no other parent, A_2, of Y which is a qualitative effect modifier for causal effect of A_1 on Y, either unconditionally or within some stratum $C=c$ of the parents of Y other than A_1 and A_2.*

A similar result clearly holds if the A_1-Y edge is of negative sign. We give the contrapositive of Proposition 9 as a corollary.

Corollary. *Suppose that some parent of Y, A_2, is a qualitative effect modifier for causal effect of another parent of Y, A_1, either unconditionally or within some stratum $C=c$ of the parents of Y other than A_1 and A_2 then A_1 can have neither a weak positive monotonic effect nor a weak negative monotonic effect on Y.*

If there are intermediate variables between A and Y then Proposition 9 can be generalized to give Proposition 10.

Proposition 10. *Suppose that all directed paths from A to Y are of positive sign (or are all of negative sign) then there exists no qualitative effect modifier Q on the directed acyclic graph for the causal effect of A on Y.*

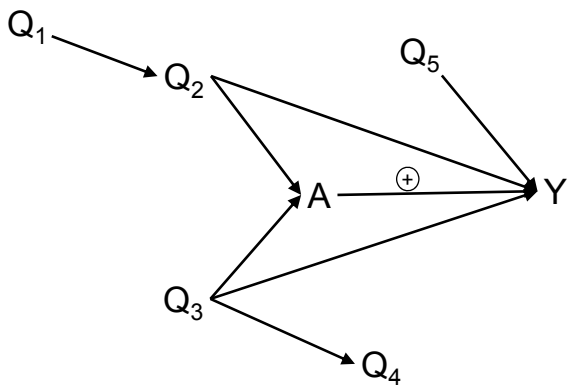

Figure 4: Example illustrating the use of Propositions 9 and 10.

Example 4. Consider the signed directed acyclic graph given in Figure 4 in which the $A - Y$ edge is of positive sign. It can be shown that any of Q_1, Q_2, Q_3, Q_4 or Q_5 can serve as effect modifiers for the causal effect of A on Y (VanderWeele and Robins, 2007). However, by Proposition 9 or 10, since A has a (weak) monotonic effect on Y, none of Q_1, Q_2, Q_3, Q_4 or Q_5 can serve as *qualitative* effect modifiers for the causal effect of A on Y. Conversely, if it is found that one of Q_1, Q_2, Q_3, Q_4 or Q_5 *is* a qualitative effect modifier for the causal effect of A on Y then the $A - Y$ edge cannot be of positive (or negative) sign.

7. Concluding Remarks

In this paper we have related weak monotonic effects to the monotonicity of certain conditional expectations in the conditioning argument and to qualitative effect modification. When the variables on a causal directed acyclic graph exhibit weak monotonic effects the results can be used to construct tests for the presence of unmeasured confounding variables. Future work could examine whether it is possible to weaken the restrictions on R in Proposition 4; another area of future research would include developing an algorithm for what relationships need systematic evaluation in order to test for particular confounding patterns; further research could also be done on the derivation of statistical tests of the type considered at the end of Section 5 for cases in which A, R and Y are not binary and for dealing with issues related to multiple testing problems.

Acknowledgments

We would like to thank the editors and three anonymous referees for helpful comments on this paper.

Appendix A. Proofs

This appendix contains the proofs for all of the results in this paper.

A.1 Proof of Proposition 1

By the definition of a non-parametric structural equation, $Y_{a,\widetilde{pa}_Y}(\omega) = f(\widetilde{pa}_Y, a, \varepsilon_Y(\omega))$ and from this the result follows.

A.2 Proof of Proposition 2

Since A has a positive monotonic effect on Y, for any $a_1 \geq a_0$ we have that $S(y|a_1, \widetilde{pa}_Y) = P(Y > y|a_1, \widetilde{pa}_Y) = P\{f(\widetilde{pa}_Y, a_1, \varepsilon_Y) > y\} \geq P\{f(\widetilde{pa}_Y, a_0, \varepsilon_Y) > y\} = P(Y > y|a_0, \widetilde{pa}_Y) = S(y|a_1, \widetilde{pa}_Y)$.

A.3 Proof of Lemma 1

For $a \geq a'$ we have $E[h(Y,A,R)|A = a, R = r] - E[h(Y,A,R)|A = a', R = r]$
$= \int_{y=-\infty}^{y=\infty} h(y,a,r)dF(y|a,r) - \int_{y=-\infty}^{y=\infty} h(y,a',r)dF(y|a',r) = \int_{y=-\infty}^{y=\infty} h(y,a,r)d\{F(y|a,r) - F(y|a',r)\} +$
$\int_{y=-\infty}^{y=\infty} \{h(y,a,r) - h(y,a',r)\}dF(y|a',r) = [h(y,a,r)\{F(y|a,r) - F(y|a',r)\}]_{y=-\infty}^{y=\infty} - \int_{y=-\infty}^{y=\infty} \{F(y|a,r) -$
$F(y|a',r)\}dh(y,a,r) + \int_{y=-\infty}^{y=\infty} \{h(y,a,r) - h(y,a',r)\}dF(y|a',r)$
$= \int_{y=-\infty}^{y=\infty} \{S(y|a,r) - S(y|a',r)\}dh(y,a,r) + \int_{y=-\infty}^{y=\infty} \{h(y,a,r) - h(y,a',r)\}dF(y|a',r)$. This final expression is non-negative since the integrands of both integrals are non-negative for $a \geq a'$.

A.4 Proof of Proposition 3

We have that $E[Y|X = x, A = a] = E[Y|\widetilde{pa}_Y, A = a]$ and since A has a (weak) positive monotonic effect on Y, we have that $S(y|a, \widetilde{pa}_Y)$ is non-decreasing in a and it follows from Lemma 1 that $E[Y|X = x, A = a] = E[Y|\widetilde{pa}_Y, A = a]$ is non-decreasing in a.

A.5 Proof of Lemma 2

We will say a path from A to B is a frontdoor path from A to B if the path begins with a directed edge with the arrowhead pointing out of A. Let Q^k and R^k be the subsets of Q and R respectively that are ancestors of $\overline{V}_k$. We will show that

$$\begin{aligned} S(v_k|a, v_1, ..., v_{k-1}, q, r) &= S(v_k|a, v_1, ..., v_{k-1}, q, r^k) \\ &= S(v_k|a, v_1, ..., v_{k-1}, q^k, r^k) = S(v_k|pa_{v_k}). \end{aligned}$$

If $R^k = R$, the first equality holds trivially. Suppose that $R^k \neq R$ so that R_m is not an ancestor of V_k. All frontdoor paths from R_m to V_k must include a collider since R_m is not an ancestor of V_k. This collider will not be in $A, V_1, ..., V_{k-1}, Q, R_1, ..., R_{m-1}$ since all these variables are non-descendents of R_m. Thus all frontdoor paths from R_m to V_k will be blocked given $A, V_1, ..., V_{k-1}, Q, R_1, ..., R_{m-1}$. All backdoor paths from R_m to V_k with an edge going into V_k will be blocked given $A, V_1, ..., V_{k-1}, Q, R_1, ..., R_{m-1}$ by pa_{V_k}; note by hypothesis it can be seen that pa_{V_k} will be contained by the variables $A, V_1, ..., V_{k-1}$, Q, R^k since there is a directed path from V_k to Y and Q includes all ancestors of Y not on directed paths from A to Y. All backdoor paths from R_m to V_k with an edge going out from V_k will be blocked given $A, Q, R_1, ..., R_{m-1}$ by hypothesis; otherwise there would be a backdoor path from R_m through V_k to Y not blocked by $A, Q, R_1, ..., R_{m-1}$. But all backdoor paths from R_m to V_k with an edge going out from V_k which are blocked by $A, Q, R_1, ..., R_{m-1}$ will also be blocked by

$A, V_1, ..., V_{k-1}, Q, R_1, ..., R_{m-1}$. This is because such a path concluding with an edge going out from V_k which is blocked by $A, Q, R_1, ..., R_{m-1}$ but not blocked by $A, V_1, ..., V_{k-1}, Q, R_1, ..., R_{m-1}$ would require that one of $V_1, ..., V_{k-1}$, say V_p, be a collider on the path or a descendent of a collider. If one of $V_1, ..., V_{k-1}$ were a collider then the path would in fact be blocked by the parents of the collider since all the parents of $V_1, ..., V_{k-1}$ are in $A, V_1, ..., V_{k-1}, Q, R_1, ..., R_{m-1}$. If one of $V_1, ..., V_{k-1}$, say V_p, were a descendent of the collider then none of the directed paths from the collider to V_p could contain nodes in $R_1, ..., R_{m-1}$ for otherwise the path would not be blocked by $A, Q, R_1, ..., R_{m-1}$; for the same reason the collider itself could not be in $R_1, ..., R_{m-1}$. But it then follows that the collider must itself be one of $V_1, ..., V_{p-1}$ since it is an ancestor of V_p with a directed path to V_p not blocked by R. However, if the collider is one of $V_1, ..., V_{p-1}$ then the path would in fact be blocked by the parents of the collider since all the parents of $V_1, ..., V_{k-1}$ are in $A, V_1, ..., V_{k-1}, Q, R_1, ..., R_{m-1}$. From this it follows that all backdoor paths from R_m to V_k with an edge going out from V_k are blocked by $A, V_1, ..., V_{k-1}, Q, R_1, ..., R_{m-1}$.

We have thus shown that V_k and R_m are d-separated given $A, V_1, ..., V_{k-1}, Q, R_1, ..., R_{m-1}$ and so

$$S(v_k|a, v_1, ..., v_{k-1}, q, r) = S(v_k|a, v_1, ..., v_{k-1}, q, r_1, ..., r_{m-1}).$$

Similarly, V_k and R_{m-1} are d-separated given $A, V_1, ..., V_{k-1}, Q, R_1, ..., R_{m-2}$ and so

$$S(v_k|a, v_1, ..., v_{k-1}, q, r_1, ..., r_{m-1}) = S(v_k|a, v_1, ..., v_{k-1}, q, r_1, ..., r_{m-2}).$$

We may carry this argument forward to get

$$S(v_k|a, v_1, ..., v_{k-1}, q, r) = S(v_k|a, v_1, ..., v_{k-1}, q, r^k).$$

All backdoor paths from V_k to $Q \backslash Q^k$ will be blocked given $A, V_1, ..., V_{k-1}, Q^k, R^k$ by pa_{v_k}. Since V_k is not a descendent of $Q \backslash Q^k$ all frontdoor paths from V_k to $Q \backslash Q^k$ will involve at least one collider which is a descendent of V_k. This collider is not in the conditioning set $A, V_1, ..., V_{k-1}, Q^k, R^k$ since this entire set consists of non-descendents of V_k and so the collider will block the frontdoor path from V_k to $Q \backslash Q^k$.

Thus V_k and $Q \backslash Q^k$ are d-separated given $A, V_1, ..., V_{k-1}, Q^k, R^k$ and so

$$S(v_k|a, v_1, ..., v_{k-1}, q, r^k) = S(v_k|a, v_1, ..., v_{k-1}, q^k, r^k).$$

Furthermore, $A, V_1, ..., V_{k-1}, Q^k, R^k$ are non-descendents of V_k and include all of the parents of V_k and so

$$S(v_k|a, v_1, ..., v_{k-1}, q^k, r^k) = S(v_k|pa_{v_k}).$$

We have thus shown as desired that

$$\begin{aligned} S(v_k|a, v_1, ..., v_{k-1}, q, r) &= S(v_k|a, v_1, ..., v_{k-1}, q, r^k) \\ &= S(v_k|a, v_1, ..., v_{k-1}, q^k, r^k) = S(v_k|pa_{v_k}). \end{aligned}$$

A.6 Proof of Lemma 3

Let $V_0 = A$ and $V_n = Y$ and let $V_1, ..., V_{n-1}$ be an ordered list of all the nodes which are not in R but which are on directed paths from A to Y such that at least one of the directed paths from each node to Y is not blocked by R. Let $\overline{V}_k = \{V_1, ..., V_k\}$. It can be shown by induction that by starting with

$n = k$ and for each k iteratively replacing by their negations the parents of V_k with negative edges into V_k suffices to obtain a graph such that all edges on all directed paths from A to Y not blocked by R have positive sign.

We can express $E[1(V_n > v)|A,Q,R]$ as

$$E[E[...E[E[1(V_n > v)|A,\overline{V}_{n-1},Q,R]|A,\overline{V}_{n-2},Q,R]|...|A,V_1,Q,R]|A,Q,R].$$

Now conditional on $A,\overline{V}_{n-1}\backslash V_i,Q,R$ we have that

$$E[1(V_n > v)|,A,\overline{V}_{n-1},Q,R]$$

is non-decreasing in v_i for $i = 1,...,n-1$ since V_i has either a weak positive monotonic effect or no effect on V_n. Thus conditional on $A,\overline{V}_{n-1}\backslash\{V_i,V_{n-1}\},Q,R$ we have that

$$E[1(V_n > v)|A,\overline{V}_{n-1},Q,R]$$

is a non-decreasing function of v_i and v_{n-1}. Furthermore, by Lemma 2 we have that $S(v_{n-1}|a,v_1,...,v_{n-2},q,r) = S(v_{n-1}|pa_{v_{n-1}})$ and so $S(v_{n-1}|a,v_1,...,v_{n-2},q,r) = S(v_{n-1}|pa_{v_{n-1}})$ is a non-decreasing in v_i for all $a,v_1,...,v_{i-1},v_{i+1},...,v_{n-2},q,r$ since V_i has either a weak positive monotonic effect or no effect on V_{n-1}. Thus by Lemma 1 we have that conditional on $A,\overline{V}_{n-2}\backslash V_i,Q,R$,

$$E[E[1(V_n > v)|A,\overline{V}_{n-1},Q,R]|A,\overline{V}_{n-2},Q,R]$$

is non-decreasing in v_i for $i = 1,...,n-2$. Carrying the argument forward, conditional on A,Q,R, we will have that

$$E[...E[E[1(V_n > v)|A,\overline{V}_{n-1},Q,R]|A,\overline{V}_{n-2},Q,R]|...|A,V_1,Q,R]$$

is a non-decreasing function of v_1 and $v_0 = a$ and since A has either a weak positive monotonic effect or no effect on V_1, $S(v_1|a,q,r) = S(v_1|pa_{v_1})$ will be non-decreasing in a and thus by Lemma 1,

$$\begin{aligned} S(y|a,q,r) &= E[1(V_n > y)|A,Q,R] \\ &= E[E[...E[E[1(V_n > y)|A,\overline{V}_{n-1},Q,R]|A,\overline{V}_{n-2},Q,R]|...|A,V_1,Q,R]|A,Q,R] \end{aligned}$$

will be non-decreasing in a.

A.7 Proof of Proposition 4

Let Q denote the set of ancestors of A or Y which are not descendents of A. Note that if for each i the backdoor paths from R_i to Y are blocked by $R_1,...,R_{i-1},A$ and X then these backdoor paths will also be blocked by $R_1,...,R_{i-1},A$ and Q since for each backdoor path from R_i to X there must be some member of $\{A\}\bigcup Q$ through which the path passes. We may thus apply Lemma 3 to conclude that $E[1(Y > y)|a,Q,r]$. Since Q blocks all backdoor paths from A to Y we have

$$\begin{aligned} S(y|a,x,r) &= E[E[1(Y > y)|a,Q,x,r]|a,x,r] \\ &= E[E[1(Y > y)|a,Q,r]|a,x,r] = E[E[1(Y > y)|a,W,r]|a,x,r] \end{aligned}$$

where W is the subset of Q which are either parents of Y or parents of a node on a directed path from A to Y. Let W' denote the subset of W for which there is a path to Y not blocked by A,X,R

then $E[E[1(Y > y)|a,W,r]|a,x,r] = E[E[1(Y > y)|a,W',r]|a,x,r]$. All backdoor paths from A to W' are blocked given R and X by X since X blocks all backdoor paths from A to Y. Any frontdoor path from A to W' will include a collider since the nodes in W' are not descendents of A. The collider cannot be in X because X includes only non-descendents of A. Suppose the collider were some node R_i; by hypothesis all backdoor paths from R_i to Y are blocked by $R_1,...,R_{i-1},A$ and X; thus the frontdoor path from A to W' would have to be blocked by $A,R_1,...,R_{i-1}$ and X for otherwise there would be a backdoor path from R_i through W' to Y not blocked by $A,R_1,...,R_{i-1}$ and X. From this it follows that every frontdoor path from A to W' must be blocked given R and X either by a collider or by a node in R or X. We have thus shown that all paths from A to W' are blocked given R and X and so W' is conditionally independent of A given R and X and so we have

$$\begin{aligned} E[E[1(Y > y)|a,W',r]|a,x,r] &= E[E[1(Y > y)|a,W',r]|x,r] \\ &= E[E[1(Y > y)|a,Q,r]|x,r]. \end{aligned}$$

We have thus shown that $S(y|a,x,r) = E[E[1(Y > y)|a,Q,r]|x,r]$. Since $E[1(Y > y)|a,Q,r]$ is non-decreasing in a for all q we also have that

$$S(y|a,x,r) = E[E[1(Y > y)|a,Q,r]|x,r]$$

is non-decreasing in a. Finally, since $S(y|a,x,r)$ is non-decreasing in a, it follows from Lemma 1 that $E[y|a,x,r]$ is also non-decreasing in a.

A.8 Proof of Proposition 5

Proposition 5 is in fact a special case of Proposition 7 with $R = \varnothing$ and $Q = \varnothing$. The proof of Proposition 7 is given below.

A.9 Proof of Proposition 6

Proposition 6 is in fact a special case of Proposition 8 with $R = \varnothing$ and $Q = \varnothing$. The proof of Proposition 8 is given below.

A.10 Proof of Proposition 7

By the law of iterated expectations,

$$\begin{aligned} &E[Y|A = a, Q = q] \\ &\quad = \sum_c E[Y|A = a, C = c, Q = q]P(C = c|A = a, Q = q) \end{aligned}$$

We have by Proposition 4 that $E[Y|A,Q,C]$ is non-decreasing in A. Let $(C_1,...,C_n)$ denote an ordered list of the variables in C. Let Q^c be variables in Q which are common causes of C and let $Q^n = Q \backslash Q^c$. Let Q_i^d be the variables in Q^c that are descendents of C_i. Let C_i^d denote the variables in C that are descendents of C_i and let $C_i^n = C \backslash \{C_i, C_i^d\}$. By Proposition 4 we have that $E[Y|A,Q,C]$ is non-decreasing in each component C_i of C by choosing for each i, A in Proposition 4 to be C_i, X in Proposition 4 to be the set $\{Q^n, Q^c \backslash Q_i^d, C_i^n\}$ and R in Proposition 4 to be the set $\{Q_i^d, C_i^d, A\}$. Furthermore,

$$P(C = c|A = a, Q = q) = \frac{P(A = a|C = c, Q = q)P(C = c|Q = q)}{P(A = a|Q = q)}$$

and so

$$P(C=c|A=1,Q=q)=\nu_q(c)P(C=c|A=0,Q=q)$$

where

$$\nu_q(c)=\frac{P(A=0|Q=q)P(A=1|C=c,Q=q)}{P(A=1|Q=q)P(A=0|C=c,Q=q)}$$

which is non-decreasing in each dimension of c since the numerator is non-decreasing in each dimension of c and the denominator is non-increasing in each dimension of c by Proposition 4 by choosing for each i, A in Proposition 4 to be C_i, X in Proposition 4 to be the set $\{Q^n, Q^c \backslash Q_i^c, C_i^n\}$ and R in Proposition 4 to be the set $\{Q_i^c, C_i^d\}$. Thus

$$\begin{aligned}
&E[Y|A=1,Q=q]\\
&=\sum_c E[Y|A=1,C=c,Q=q]P(C=c|A=1,Q=q)\\
&\geq \sum_c E[Y|A=0,C=c,Q=q]P(C=c|A=1,Q=q)\\
&=\sum_c E[Y|A=0,C=c,Q=q]\nu_q(c)P(C=c|A=0,Q=q)\\
&\geq \sum_c E[Y|A=0,C=c,Q=q]P(C=c|A=0,Q=q)\\
&=E[Y|A=0,Q=q].
\end{aligned}$$

The second inequality holds because by an argument similar to that above $E[Y|A=0,Q=q,C=c]$ is non-decreasing in each dimension of c and $P(C=c|A=1,Q=q)=\nu_q(c)P(C=c|A=0,Q=q)$ weights more heavily higher values of each dimension of c than does $P(C=c|A=0,Q=q)$ since $\nu_q(c)$ is non-decreasing in each dimension of c. Thus $E[Y|A=a,Q=q]$ is non-decreasing in a.

A.11 Proof of Proposition 8

By the law of iterated expectations we have that

$$\begin{aligned}
&E[A|Y=y,Q=q]=\sum_c E[A|Y=y,C=c,Q=q]P(C=c|Y=y,Q=q)\\
&=\sum_{c,a} aP(A=a|Y=y,C=c,Q=q)P(C=c|Y=y,Q=q)\\
&=\sum_{c,a} a\frac{P(Y=y,A=a,C=c|Q=q)}{P(Y=y,C=c|Q=q)}P(C=c|Y=y,Q=q)\\
&=\sum_{c,a} a\frac{P(Y=y|A=a,C=c,Q=q)}{P(Y=y|Q=q)}P(A=a,C=c|Q=q)\\
&=E_{C,A}[A\frac{P(Y=y|A,C,Q=q)}{P(Y=y|Q=q)}|Q=q].
\end{aligned}$$

As in the proof of Proposition 7, we have by Proposition 4 we have that conditional on and $Q=q$, $\frac{P(Y=1|A,C,Q=q)}{P(Y=1|Q=q)}$ is a non-decreasing function of A and of each dimension of C. Similarly, $\frac{P(Y=0|A,C,Q=q)}{P(Y=0|Q=q)}$ is a non-increasing function of A and each dimension of C. Over c and a, conditional on and $Q=q$, $\frac{P(Y=y|A=a,C=c,Q=q)}{P(Y=y|Q=q)}$ is a weight function that sums to 1, that is, $E_{C,A}[\frac{P(Y=y|A=a,C=c,Q=q)}{P(Y=y|Q=q)}]=$

$\frac{P(Y=y|Q=q)}{P(Y=y|Q=q)} = 1$. Furthermore, by Proposition 4, $S(a|c,q)$ is non-decreasing in c and we thus have that

$$\begin{aligned} E[A|Y=1,Q=q] &= E_{C,A}[A\frac{P(Y=1|A,C,Q=q)}{P(Y=1|Q=q)}|Q=q] \\ &\geq E_{C,A}[A\frac{P(Y=0|A,C,Q=q)}{P(Y=0|Q=q)}|Q=q] \\ &= E[A|Y=0,Q=q] \end{aligned}$$

and so $E[A|Y,Q]$ is non-decreasing in Y.

A.12 Proof of Proposition 9

Note that by Proposition 3 above if A_1 has a weak positive monotonic effect on Y then $E[Y|A_1 = a_1, A_2 = a_2, C = c]$ must be non-decreasing in a_1 and if A_1 has a weak negative monotonic effect on Y then $E[Y|A_1 = a_1, A_2 = a_2, C = c]$ must be non-increasing in a_1. Since $(Y \coprod A_1|\{A_2,C\})_{G_{\underline{E_1}}}$ where $G_{\underline{E_1}}$ is the original directed acyclic graph G with all edges emanating from A_1 removed, we have $Y_{A_1=a} \coprod A_1|\{A_2,C\}$ (Pearl, 1995). Thus $E[Y_{A_1=a_1}|A_2 = a_2, C = c] = E[Y|A_1 = a_1, A_2 = a_2, C = c]$ and so if A_2 is a qualitative effect modifier for the causal effect of A_1 on Y for stratum $C = c$ then we must two values of A_1, a_1^* and a_1^{**}, and two levels of A_2, a_2' and a_2'', such that $E[Y|A_1 = a_1^{**}, A_2 = a_2'', C = c] - E[Y|A_1 = a_1^*, A_2 = a_2'', C = c] < 0$ and $E[Y|A_1 = a_1^{**}, A_2 = a_2', C = c] - E[Y|A_1 = a_1^*, A_2 = a_2', C = c] > 0$. Either $a_1^{**} > a_1^*$ or $a_1^{**} < a_1^*$. Consider the first case (the second is analogous) then since $E[Y|A_1 = a_1^{**}, A_2 = a_2'', C = c] - E[Y|A_1 = a_1^*, A_2 = a_2'', C = c] < 0$, A_1 does not have a weak positive monotonic effect on Y and since $E[Y|A_1 = a_1^{**}, A_2 = a_2', C = c] - E[Y|A_1 = a_1^*, A_2 = a_2', C = c] > 0$, A_1 does not have a weak negative monotonic effect on Y. Now if A_2 is a qualitative effect modifier for the causal effect of A_1 unconditionally then we must have two values of A_1, a_1^* and a_1^{**}, and two levels of A_2, a_2' and a_2'', such that $E[Y_{A_1=a_1^{**}}|A_2 = a_2''] - E[Y_{A_1=a_1^*}|A_2 = a_2''] < 0$ and $E[Y_{A_1=a_1^{**}}|A_2 = a_2'] - E[Y_{A_1=a_1^*}|A_2 = a_2'] > 0$. Once again either $a_1^{**} > a_1^*$ or $a_1^{**} < a_1^*$. We will consider the first case (the second is analogous). We thus have that $\sum_c E[Y|A_1 = a_1^{**}, A_2 = a_2'', C = c]P(C = c|A_2 = a_2'') = \sum_c E[Y_{A_1=a_1^{**}}|A_2 = a_2'', C = c]P(C = c|A_2 = a_2'') = E[Y_{A_1=a_1^{**}}|A_2 = a_2''] < E[Y_{A_1=a_1^*}|A_2 = a_2''] = \sum_c E[Y_{A_1=a_1^*}|A_2 = a_2'', C = c]P(C = c|A_2 = a_2'') = \sum_c E[Y|A_1 = a_1^*, A_2 = a_2'', C = c]P(C = c|A_2 = a_2'')$ and so A_1 cannot have a weak positive monotonic effect on Y and similarly, $\sum_c E[Y|A_1 = a_1^{**}, A_2 = a_2', C = c]P(C = c|A_2 = a_2') = \sum_c E[Y_{A_1=a_1^{**}}|A_2 = a_2', C = c]P(C = c|A_2 = a_2') = E[Y_{A_1=a_1^{**}}|A_2 = a_2'] > E[Y_{A_1=a_1^*}|A_2 = a_2'] = \sum_c E[Y_{A_1=a_1^*}|A_2 = a_2', C = c]P(C = c|A_2 = a_2') = \sum_c E[Y|A_1 = a_1^*, A_2 = a_2', C = c]P(C = c|A_2 = a_2')$ and so A_1 cannot have a weak negative monotonic effect on Y.

A.13 Proof of Proposition 10

We prove the Theorem for weak positive monotonic effects. The proof for weak negative monotonic effects is similar. Let C denote all non-descendents of A which are either parents of Y or parents of a node on a directed path between A and Y. By the law of iterated expectations we have $E[Y_{A=a_1}|Q=q] - E[Y_{A=a_0}|Q=q] = \sum_c E[Y_{A=a_1}|C=c,Q=q]P(C=c|Q=q) - \sum_c E[Y_{A=a_0}|C=c,Q=q]P(C=c|Q=q)$. We will show that this latter expression is equal to $\sum_c E[Y_{A=a_1}|C=c]P(C=c|Q=q) - \sum_c E[Y_{A=a_0}|C=c]P(C=c|Q=q)$. By Theorem 3 of Pearl (1995) it suffices

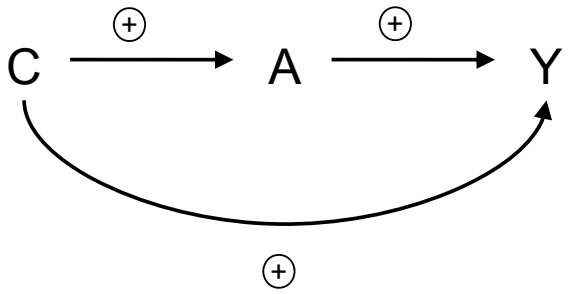

Figure 5: Directed acyclic graph illustrating counterexamples to Propositions 5 and 6 when A is not binary.

to show that $(Y \coprod Q|C,A)_{G_{\overline{A}}}$ where $G_{\overline{A}}$ denotes the graph obtained by deleting from the original directed acyclic graph all arrows pointing into A. Any front door path from Y to Q in $G_{\overline{A}}$ will be blocked by a collider. Any backdoor path from Y to Q in $G_{\overline{A}}$ will be blocked by C. We thus have that $E[Y_{A=a_1}|Q=q]-E[Y_{A=a_0}|Q=q]=\sum_c E[Y_{A=a_1}|C=c]P(C=c|Q=q)-\sum_c E[Y_{A=a_0}|C=c]P(C=c|Q=q)$. Since C will block all backdoor paths from A to Y we have by the backdoor path adjustment theorem $\sum_c E[Y|C=c,A=a_1]P(C=c|Q=q)-\sum_c E[Y|C=c,A=a_0]P(C=c|Q=q)=\sum_c\{E[Y|C=c,A=a_1]-E[Y|C=c,A=a_0]\}P(C=c|Q=q)$. If there were a qualitative effect modifier Q for the causal effect of A on Y then there would exist a value q_0 such that $E[Y_{A=a_1}|Q=q_0]-E[Y_{A=a_0}|Q=q_0]<0$. But since all paths between A and Y are of positive sign and since C blocks all backdoor paths from A to Y we have by Proposition 4 that $E[Y|C=c,A=a]$ is non-decreasing in a and so $E[Y_{A=a_1}|Q=q_0]-E[Y_{A=a_0}|Q=q_0]=\sum_c\{E[Y|C=c,A=a_1]-E[Y|C=c,A=a_0]\}P(C=c|Q=q_0)\geq 0$.

Appendix B. Counterexamples

This appendix contains the details of the counterexamples mentioned in the body of the paper.

B.1 Counterexample 1

Consider the directed acyclic graph given in Figure 5. In this example C and Y are binary and A is ternary. Suppose that $C\sim Ber(0.5)$, $\varepsilon_A\sim Ber(0.5)$ and that $P(A=0|\varepsilon_A=0)=1$ and $P(A=C+1|\varepsilon_A=1)=1$. Suppose also that $P(Y=1|A=2)=1$ and that if $P(Y=C|A=0)=1$ and $P(Y=C|A=1)=1$. Clearly then C has a positive monotonic effect on A and on Y and A has a positive monotonic effect on Y and so A and Y are positively monotonically associated. However, we have that $E[Y|A=1]=E[C|A=1]=0*P(C=1|A=1)=0$ but $E[Y|A=0]=E[C|A=0]=1*P(C=1|A=0)+0*P(C=0|A=0)=1/2$.

B.2 Counterexample 2

Consider again the directed acyclic graph given in Figure 5. In this example we will assume that C and A are binary and that Y is ternary. Suppose that $C\sim Ber(0.5)$ and that ε_A takes on the values 0, 1 and 2, each with probability $1/3$. Suppose also that $P(A=0|\varepsilon_A=0)=1$, $P(A=C|\varepsilon_A=1)=1$ and

$P(A = 1|\varepsilon_A = 2) = 1$. Suppose further that $P(Y = 0|C = 0) = 1$ and if $P(Y = A+1|C = 1)$. Clearly then C has a positive monotonic effect on A and on Y and A has a positive monotonic effect on Y and so A and Y are positively monotonically associated. However, we have that $E[A|Y = 1] = 0$ but $E[A|Y = 0] = E[A|C = 0] = 1/3$.

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