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A Simpler Approach to Matrix Completion

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Abstract

This paper provides the best bounds to date on the number of randomly sampled entries required to reconstruct an unknown low-rank matrix. These results improve on prior work by Candès and Recht (2009), Candès and Tao (2009), and Keshavan et al. (2009). The reconstruction is accomplished by minimizing the nuclear norm, or sum of the singular values, of the hidden matrix subject to agreement with the provided entries. If the underlying matrix satisfies a certain incoherence condition, then the number of entries required is equal to a quadratic logarithmic factor times the number of parameters in the singular value decomposition. The proof of this assertion is short, self contained, and uses very elementary analysis. The novel techniques herein are based on recent work in quantum information theory.

Keywords: matrix completion, low-rank matrices, convex optimization, nuclear norm minimization, random matrices, operator Chernoff bound, compressed sensing

1. Introduction

Recovering a low-rank matrix from a partial sampling of its entries is a recurring problem in collaborative filtering (Rennie and Srebro, 2005; Koren et al., 2009) and dimensionality reduction (Weinberger and Saul, 2006; So and Ye, 2007). Estimating of low-rank models also arise in embedding problems (Linial et al., 1995) and multi-class learning (Argyriou et al., 2008; Obozinski et al., 2009). While a variety of heuristics have been developed across many disciplines, the general problem of finding the lowest rank matrix satisfying equality constraints is NP-hard. All known algorithms which can compute the lowest rank solution for all instances require time at least exponential in the dimensions of the matrix in both theory and practice (Chistov and Grigoriev, 1984).

In sharp contrast to such worst case pessimism, Candès and Recht (2009) showed that most low-rank matrices could be recovered from most sufficiently large sets of entries by computing the matrix of minimum *nuclear norm* that agreed with the provided entries, and furthermore the revealed set of entries could comprise a vanishing fraction of the entire matrix. The nuclear norm is equal to the sum of the singular values of a matrix and is the best convex lower bound of the rank function on the set of matrices whose singular values are all bounded by 1. The intuition behind this heuristic is that whereas the rank function counts the number of nonvanishing singular values, the nuclear norm sums their amplitude, much like how the ℓ_1 norm is a useful surrogate for counting the number of nonzeros in a vector. Moreover, the nuclear norm can be minimized subject to equality constraints via semidefinite programming.

Nuclear norm minimization had long been observed to produce very low-rank solutions in practice (see, for example, Beck and D'Andrea, 1998; Fazel, 2002; Fazel et al., 2001; Srebro, 2004;

Mesbahi and Papavassilopoulos, 1997), but only very recently was there any theoretical basis for when it produced the minimum rank solution. The first paper to provide such foundations was Recht et al. (2010), where the authors developed probabilistic techniques to study average case behavior and showed that the nuclear norm heuristic could solve most instances of the linearly-constrained rank-minimization problem assuming the number of linear constraints was sufficiently large. The results in Recht et al. (2010) inspired a groundswell of interest in theoretical guarantees for rank minimization, and these results lay the foundation for Candès and Recht (2009). Candès and Recht's bounds were subsequently improved by Candès and Tao (2009) and Keshavan et al. (2009) to show that one could, in special cases, reconstruct a low-rank matrix by observing a set of entries of size at most a polylogarithmic factor larger than the intrinsic dimension of the variety of rank r matrices.

This paper sharpens the results in Candès and Tao (2009) and Keshavan et al. (2009) to provide a bound on the number of entries required to reconstruct a low-rank matrix which is optimal up to a small numerical constant and one logarithmic factor. The main theorem makes minimal assumptions about the low-rank matrix of interest. Moreover, the proof is very short and relies on mostly elementary analysis.

In order to precisely state the main result, we need one definition. Candès and Recht observed that it is impossible to recover a matrix which is equal to zero in nearly all of its entries unless all of the entries of the matrix are observed (consider, for example, the rank one matrix which is equal to 1 in one entry and zeros everywhere else). In other words, the matrix cannot be mostly equal to zero on the observed entries. This motivated the following definition

Definition 1 *Let U be a subspace of $\mathbb{R}^n$ of dimension r and $\boldsymbol{P}_U$ be the orthogonal projection onto U. Then the* coherence *of U (vis-à-vis the standard basis $(\boldsymbol{e}_i)$) is defined to be*

$$\mu(U) \equiv \frac{n}{r} \max_{1 \leq i \leq n} \|\boldsymbol{P}_U \boldsymbol{e}_i\|^2.$$

Note that for any subspace, the smallest $\mu(U)$ can be is 1, achieved, for example, if U is spanned by vectors whose entries all have magnitude $1/\sqrt{n}$. The largest possible value for $\mu(U)$ is n/r which would correspond to any subspace that contains a standard basis element. If a matrix has row and column spaces with low coherence, then each entry can be expected to provide about the same amount of information.

Recall that the *nuclear norm* of an $n_1 \times n_2$ matrix $\boldsymbol{X}$ is the sum of the singular values of $\boldsymbol{X}$, $\|\boldsymbol{X}\|_* = \sum_{k=1}^{\min\{n_1,n_2\}} \sigma_k(\boldsymbol{X})$, where, here and below, $\sigma_k(\boldsymbol{X})$ denotes the kth largest singular value of $\boldsymbol{X}$. The main result of this paper is the following

Theorem 2 *Let $\boldsymbol{M}$ be an $n_1 \times n_2$ matrix of rank r with singular value decomposition $\boldsymbol{U\Sigma V}^*$. Without loss of generality, impose the conventions $n_1 \leq n_2$, $\boldsymbol{\Sigma}$ is $r \times r$, $\boldsymbol{U}$ is $n_1 \times r$ and $\boldsymbol{V}$ is $n_2 \times r$. Assume that*

A0 *The row and column spaces have coherences bounded above by some positive μ_0.*

A1 *The matrix $\boldsymbol{UV}^*$ has a maximum entry bounded by $\mu_1\sqrt{r/(n_1 n_2)}$ in absolute value for some positive μ_1.*

Suppose m entries of $\boldsymbol{M}$ are observed with locations sampled uniformly at random. Then if

$$m \geq 32 \max\{\mu_1^2, \mu_0\}\, r(n_1+n_2)\, \beta \log^2(2n_2) \tag{1}$$

for some $\beta > 1$, *the minimizer to the problem*

$$\begin{array}{ll} \textit{minimize} & \|\boldsymbol{X}\|_* \\ \textit{subject to} & X_{ij} = M_{ij} \quad (i,j) \in \Omega. \end{array} \tag{2}$$

is unique and equal to $\boldsymbol{M}$ *with probability at least* $1 - 6\log(n_2)(n_1+n_2)^{2-2\beta} - n_2^{2-2\beta^{1/2}}$.

The assumptions **A0** and **A1** were introduced in Candès and Recht (2009). Both μ_0 and μ_1 may depend on r, n_1, or n_2. Moreover, note that $\mu_1 \leq \mu_0\sqrt{r}$ by the Cauchy-Schwarz inequality. As shown in Candès and Recht (2009), both subspaces selected from the uniform distribution and spaces constructed as the span of singular vectors with bounded entries are not only incoherent with the standard basis, but also obey **A1** with high probability for values of μ_1 at most logarithmic in n_1 and/or n_2. Applying this theorem to the models studied in Section 2 of Candès and Recht (2009), we find that there is a numerical constant c_u such that $c_u r(n_1+n_2)\log^5(n_2)$ entries are sufficient to reconstruct a rank r matrix whose row and column spaces are sampled from the Haar measure on the Grassmann manifold. If $r > \log(n_2)$, the number of entries can be reduced to $c_u r(n_1+n_2)\log^4(n_2)$. Similarly, there is a numerical constant c_i such that $c_i\mu_0^2 r(n_1+n_2)\log^3(n_2)$ entries are sufficient to recover a matrix of arbitrary rank r whose singular vectors have entries with magnitudes bounded by $\sqrt{\mu_0/n_1}$.

Theorem 2 greatly improves upon prior results. First of all, it has the weakest assumptions on the matrix to be recovered. In addition to assumption **A1**, Candès and Tao (2009) require a “strong incoherence condition” which is considerably more restrictive than the assumption **A0** in Theorem 2. Many of their results also require restrictions on the rank of $\boldsymbol{M}$, and their bounds depend superlinearly on μ_0. Keshavan et al. (2009) require the matrix rank to be no more than $\log(n_2)$, and require bounds on the maximum magnitude of the entries in $\boldsymbol{M}$ and the ratios $\sigma_1(\boldsymbol{M})/\sigma_r(\boldsymbol{M})$ and n_2/n_1. Theorem 2 makes no such assumptions about the rank, aspect ratio, nor condition number of $\boldsymbol{M}$. Moreover, (1) has a smaller log factor than Candès and Tao (2009), and features numerical constants that are both explicit and small.

Also note that there is not much room for improvement in the bound for m. It is a consequence of the coupon collector’s problem that at least $n_2\log n_2$ uniformly sampled entries are necessary just to guarantee that at least one entry in every row and column is observed with high probability. In addition, rank r matrices have $r(n_1+n_2-r)$ parameters, a fact that can be verified by counting the number of degrees of freedom in the singular value decomposition. Interestingly, Candès and Tao (2009) showed that $C\mu_0 n_2 r\log(n_2)$ entries were *necessary* for completion when the entries are sampled uniformly at random. Hence, (1) is optimal up to a small numerical constant times $\log(n_2)$.

Most importantly, the proof of Theorem 2 is short and straightforward. Candès and Recht employed sophisticated tools from the study of random variables on Banach spaces including decoupling tools and powerful moment inequalities for the norms of random matrices. Candès and Tao rely on intricate moment calculations spanning over 30 pages. The present work only uses basic matrix analysis, elementary large deviation bounds, and a noncommutative version of Bernstein’s Inequality proven here in the Appendix.

The proof of Theorem 2 is adapted from a recent paper by Gross et al. (2010) in quantum information which considered the problem of reconstructing the density matrix of a quantum ensemble using as few measurements as possible. Their work extended results from Candès and Recht (2009) to the quantum regime by using special algebraic properties of quantum measurements. Their proof followed a methodology analogous to the approach of Candès and Recht but had three main differences: they used a sampling with replacement model as a proxy for uniform sampling, deployed a powerful noncommutative Chernoff bound developed by Ahlswede and Winter (2002) for use in quantum information theory, and devised a simplified appeal to convex duality to guarantee exact recovery. In this paper, I adapt these strategies from Gross et al. (2010) to the matrix completion problem. In Section 3 I show how the sampling with replacement model bounds probabilities in the uniform sampling model, and present very short proofs of some of the main results in Candès and Recht (2009). Surprisingly, this yields a simple proof of Theorem 2, provided in Section 4, which has the least restrictive assumptions of any assertion proven thus far.[1]

2. Preliminaries and Notation

Before continuing, let us survey the notations used throughout the paper. I closely follow the conventions established in Candès and Recht (2009), and invite the reader to consult this reference for a more thorough discussion of the matrix completion problem and the associated convex geometry. A thorough introduction to the necessary matrix analysis used in this paper can be found in Recht et al. (2010).

Matrices are bold capital, vectors are bold lowercase and scalars or entries are not bold. For example, $\boldsymbol{X}$ is a matrix, and X_{ij} its (i,j)th entry. Likewise $\boldsymbol{x}$ is a vector, and x_i its ith component. If $\boldsymbol{u}_k \in \mathbb{R}^n$ for $1 \leq k \leq d$ is a collection of vectors, $[\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d]$ will denote the $n \times d$ matrix whose kth column is $\boldsymbol{u}_k$. $\boldsymbol{e}_k$ will denote the kth standard basis vector in $\mathbb{R}^d$, equal to 1 in component k and 0 everywhere else. The dimension of $\boldsymbol{e}_k$ will always be clear from context. $\boldsymbol{X}^*$ and $\boldsymbol{x}^*$ denote the transpose of matrices $\boldsymbol{X}$ and vectors $\boldsymbol{x}$ respectively.

A variety of norms on matrices will be discussed. The spectral norm of a matrix is denoted by $\|\boldsymbol{X}\|$. The Euclidean inner product between two matrices is $\langle \boldsymbol{X}, \boldsymbol{Y} \rangle = \mathrm{Tr}(\boldsymbol{X}^*\boldsymbol{Y})$, and the corresponding Euclidean norm, called the Frobenius or Hilbert-Schmidt norm, is denoted $\|\boldsymbol{X}\|_F$. That is, $\|\boldsymbol{X}\|_F = \langle \boldsymbol{X}, \boldsymbol{X} \rangle^{1/2}$. The nuclear norm of a matrix $\boldsymbol{X}$ is $\|\boldsymbol{X}\|_*$. The maximum entry of $\boldsymbol{X}$ (in absolute value) is denoted by $\|\boldsymbol{X}\|_\infty \equiv \max_{ij} |X_{ij}|$. For vectors, the only norm applied is the usual Euclidean ℓ_2 norm, simply denoted as $\|\boldsymbol{x}\|$.

Linear transformations that act on matrices will be denoted by calligraphic letters. In particular, the identity operator will be denoted by $\mathcal{I}$. The spectral norm (the top singular value) of such an operator will be denoted by $\|\mathcal{A}\| = \sup_{\boldsymbol{X}:\|\boldsymbol{X}\|_F \leq 1} \|\mathcal{A}(\boldsymbol{X})\|_F$.

Fix once and for all a matrix $\boldsymbol{M}$ obeying the assumptions of Theorem 2. Let $\boldsymbol{u}_k$ (respectively $\boldsymbol{v}_k$) denote the kth column of $\boldsymbol{U}$ (respectively $\boldsymbol{V}$). Set $U \equiv \mathrm{span}\,(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r)$, and $V \equiv \mathrm{span}\,(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r)$. Also assume, without loss of generality, that $n_1 \leq n_2$. It is convenient to introduce the orthogonal decomposition $\mathbb{R}^{n_1 \times n_2} = T \oplus T^\perp$ where T is the linear space spanned by elements of the form $\boldsymbol{u}_k \boldsymbol{y}^*$ and $\boldsymbol{x}\boldsymbol{v}_k^*$, $1 \leq k \leq r$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are arbitrary, and $T^\perp$ is its orthogonal complement. $T^\perp$ is the subspace of matrices spanned by the family $(\boldsymbol{x}\boldsymbol{y}^*)$, where $\boldsymbol{x}$ (respectively $\boldsymbol{y}$) is any vector orthogonal to U (respectively V).

The orthogonal projection $\mathcal{P}_T$ onto T is given by

$$\mathcal{P}_T(\boldsymbol{Z}) = \boldsymbol{P}_U \boldsymbol{Z} + \boldsymbol{Z} \boldsymbol{P}_V - \boldsymbol{P}_U \boldsymbol{Z} \boldsymbol{P}_V, \tag{3}$$

1. Shortly after the appearance of a preprint of this manuscript, Gross (2011) announced a far reaching generalization of the techniques in Gross et al. (2010), providing bounds on recovering low-rank matrices in almost any basis. This work is more general than the work presented here, but the present paper achieves tighter constants and bounds and work directly with non-Hermitian matrices. The interested reader should consult Gross (2011) for more details.

where $\boldsymbol{P}_U$ and $\boldsymbol{P}_V$ are the orthogonal projections onto U and V respectively. Note here that while $\boldsymbol{P}_U$ and $\boldsymbol{P}_V$ are matrices, $\mathcal{P}_T$ is a linear operator mapping matrices to matrices. The orthogonal projection onto $T^\perp$ is given by

$$\mathcal{P}_{T^\perp}(\boldsymbol{Z}) = (I - \mathcal{P}_T)(\boldsymbol{Z}) = (\boldsymbol{I}_{n_1} - \boldsymbol{P}_U)\boldsymbol{Z}(\boldsymbol{I}_{n_2} - \boldsymbol{P}_V)$$

where $\boldsymbol{I}_d$ denotes the $d \times d$ identity matrix. It follows from the definition (3) of $\mathcal{P}_T$ that

$$\mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*) = (\boldsymbol{P}_U\boldsymbol{e}_a)\boldsymbol{e}_b^* + \boldsymbol{e}_a(\boldsymbol{P}_V\boldsymbol{e}_b)^* - (\boldsymbol{P}_U\boldsymbol{e}_a)(\boldsymbol{P}_V\boldsymbol{e}_b)^*.$$

This gives

$$\|\mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*)\|_F^2 = \langle \mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*), \boldsymbol{e}_a\boldsymbol{e}_b^* \rangle = \|\boldsymbol{P}_U\boldsymbol{e}_a\|^2 + \|\boldsymbol{P}_V\boldsymbol{e}_b\|^2 - \|\boldsymbol{P}_U\boldsymbol{e}_a\|^2\,\|\boldsymbol{P}_V\boldsymbol{e}_b\|^2.$$

Since $\|\boldsymbol{P}_U\boldsymbol{e}_a\|^2 \le \mu(U)r/n_1$ and $\|\boldsymbol{P}_V\boldsymbol{e}_b\|^2 \le \mu(V)r/n_2$,

$$\|\mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*)\|_F^2 \le \max\{\mu(U),\mu(V)\} r \frac{n_1+n_2}{n_1n_2} \le \mu_0 r \frac{n_1+n_2}{n_1n_2}. \tag{4}$$

I will make frequent use of this calculation throughout the sequel.

3. Sampling with Replacement

As discussed above, the main contribution of this work is an analysis of uniformly sampled sets of entries via the study of a sampling with replacement model. All of the previous work (e.g., Candès and Recht, 2009; Candès and Tao, 2009; Keshavan et al., 2009) studied a Bernoulli sampling model as a proxy for uniform sampling. There, each entry was revealed independently with probability equal to p. In all of these results, the theorem statements concerned sampling sets of m entries uniformly, but it was shown that probability of failure under Bernoulli sampling with $p = \frac{m}{n_1n_2}$ closely approximated the probability of failure under uniform sampling. The present work will analyze the situation where each entry index is sampled independently from the uniform distribution on $\{1,\dots,n_1\} \times \{1,\dots,n_2\}$. This modification of the sampling model gives rise to all of the simplifications below.

It would appear that sampling with replacement is not suitable for analyzing matrix completion as one might encounter duplicate entries. However, just as is the case with Bernoulli sampling, bounding the likelihood of error when sampling with replacement allows us to bound the probability of the nuclear norm heuristic failing under uniform sampling.

Proposition 3 *The probability that the nuclear norm heuristic fails when the set of observed entries is sampled uniformly from the collection of sets of size m is less than or equal to the probability that the heuristic fails when m entries are sampled independently with replacement.*

Proof The proof follows the argument in Section II.C of Candés et al. (2006). Let Ω' be a collection of m entries, each sampled independently from the uniform distribution on $\{1,\dots,n_1\} \times \{1,\dots,n_2\}$. Let Ω_k denote a set of entries of size k sampled uniformly from all collections of entries of size k.

It follows that

$$\begin{aligned}\mathbb{P}(\text{Failure}(\Omega')) &= \sum_{k=0}^{m} \mathbb{P}(\text{Failure}(\Omega') \mid |\Omega'| = k)\, \mathbb{P}(|\Omega'| = k) \\ &= \sum_{k=0}^{m} \mathbb{P}(\text{Failure}(\Omega_k))\, \mathbb{P}(|\Omega'| = k) \\ &\geq \mathbb{P}(\text{Failure}(\Omega_m)) \sum_{k=0}^{m} \mathbb{P}(|\Omega'| = k) = \mathbb{P}(\text{Failure}(\Omega_m)) \,.\end{aligned}$$

Where the inequality follows because $\mathbb{P}(\text{Failure}(\Omega_m)) \geq \mathbb{P}(\text{Failure}(\Omega_{m'}))$ if $m \leq m'$. That is, the probability decreases as the number of entries revealed is increased. ■

Surprisingly, changing the sampling model makes most of the theorems from Candès and Recht (2009) simple consequences of a noncommutative variant of Bernstein's Inequality.

Theorem 4 (Noncommutative Bernstein Inequality) *Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_L$ be independent zero-mean random matrices of dimension $d_1 \times d_2$. Suppose $\rho_k^2 = \max\{\|\mathbb{E}[\boldsymbol{X}_k \boldsymbol{X}_k^*]\|, \|\mathbb{E}[\boldsymbol{X}_k^* \boldsymbol{X}_k]\|\}$ and $\|\boldsymbol{X}_k\| \leq M$ almost surely for all k. Then for any $\tau > 0$,*

$$\mathbb{P}\left[\left\|\sum_{k=1}^{L} \boldsymbol{X}_k\right\| > \tau\right] \leq (d_1 + d_2) \exp\left(\frac{-\tau^2/2}{\sum_{k=1}^{L} \rho_k^2 + M\tau/3}\right) .$$

Note that in the case that $d_1 = d_2 = 1$, this is precisely the two sided version of the standard Bernstein Inequality. When the $\boldsymbol{X}_k$ are diagonal, this bound is the same as applying the standard Bernstein Inequality and a union bound to the diagonal of the matrix summation. Furthermore, observe that the right hand side is less than $(d_1 + d_2) \exp(-\frac{3}{8}\tau^2 / (\sum_{k=1}^{L} \rho_k^2))$ as long as $\tau \leq \frac{1}{M} \sum_{k=1}^{L} \rho_k^2$. This condensed form of the inequality will be used exclusively throughout. Theorem 4 is a corollary of an Chernoff bound for finite dimensional operators developed by Ahlswede and Winter (2002). A similar inequality for symmetric i.i.d. matrices is proposed in Gross et al. (2010). The proof is provided in the Appendix.

Let us now record two theorems, proven for the Bernoulli model in Candès and Recht (2009), that admit very simple proofs in the sampling with replacement model. The theorem statements requires some additional notation. Let $\Omega = \{(a_k, b_k)\}_{k=1}^{l}$ be a collection of indices sampled uniformly with replacement. Set $\mathcal{R}_\Omega$ to be the operator

$$\mathcal{R}_\Omega(\boldsymbol{Z}) = \sum_{k=1}^{|\Omega|} \langle \boldsymbol{e}_{a_k} \boldsymbol{e}_{b_k}^*, \boldsymbol{Z} \rangle \boldsymbol{e}_{a_k} \boldsymbol{e}_{b_k}^* \,.$$

Note that the (i, j)th component of $\mathcal{R}_\Omega(\boldsymbol{X})$ is zero unless $(i, j) \in \Omega$. For $(i, j) \in \Omega$, $\mathcal{R}_\Omega(\boldsymbol{X})$ is equal to X_{ij} times the multiplicity of $(i, j) \in \Omega$. Unlike in previous work on matrix completion, $\mathcal{R}_\Omega$ is not a projection operator if there are duplicates in Ω. Nonetheless, this does not adversely affect the argument, and $\mathcal{R}_\Omega(\boldsymbol{X}) = 0$ if and only if $X_{ab} = 0$ for all $(a, b) \in \Omega$. Moreover, we can show that the maximum duplication of any entry is always less than $\frac{8}{3}\log(n_2)$ with very high probability.

Proposition 5 *With probability at least $1 - n_2^{2-2\beta}$, the maximum number of repetitions of any entry in Ω is less than $\frac{8}{3}\beta \log(n_2)$ for $n_2 \geq 9$ and $\beta > 1$.*

Proof This assertion can be proven by applying a standard Chernoff bound for the Bernoulli distribution. Note that for a fixed entry, the probability it is sampled more than t times is equal to the probability of more than t heads occurring in a sequence of m tosses where the probability of a head is $\frac{1}{n_1 n_2}$. This probability can be upper bounded by

$$\mathbb{P}[\text{more than } t \text{ heads in } m \text{ trials}] \le \left(\frac{m}{n_1 n_2 t}\right)^t \exp\left(t - \frac{m}{n_1 n_2}\right)$$

(see Hagerup and Rüb, 1990, for example). Applying the union bound over all of the $n_1 n_2$ entries and the fact that $\frac{m}{n_1 n_2} < 1$, we have

$$\begin{aligned}&\mathbb{P}[\text{any entry is selected more than } \tfrac{8}{3}\beta\log(n_2) \text{ times}]\\ &\qquad\le n_1 n_2 \left(\tfrac{8}{3}\beta\log(n_2)\right)^{-\frac{8}{3}\beta\log(n_2)} \exp\left(\tfrac{8}{3}\beta\log(n_2)\right)\\ &\qquad\le n_2^{2-2\beta}\end{aligned}$$

when $n_2 \ge 9$. ∎

This application of the Chernoff bound is very crude, and much tighter bounds can be derived using more careful analysis. For example in Gonnet (1981), the maximum oversampling is shown to be bounded by $O(\frac{\log(n_2)}{\log\log(n_2)})$. For our purposes here, the loose upper bound provided by Proposition 5 will be more than sufficient.

In addition to this bound on the norm of $\mathcal{R}_\Omega$, the following theorem asserts that the operator $\mathcal{P}_T \mathcal{R}_\Omega \mathcal{P}_T$ is also very close to an isometry on T if the number of sampled entries is sufficiently large. This result is analgous to the Theorem 4.1 in Candès and Recht (2009) for the Bernoulli model, whose proof uses several powerful theorems from the study of probability in Banach spaces. Here, one only needs to compute a few low order moments and then apply Theorem 4.

Theorem 6 *Suppose Ω is a set of entries of size m sampled independently and uniformly with replacement. Then for all $\beta > 1$,*

$$\frac{n_1 n_2}{m}\left\|\mathcal{P}_T \mathcal{R}_\Omega \mathcal{P}_T - \frac{m}{n_1 n_2}\mathcal{P}_T\right\| \le \sqrt{\frac{16\mu_0 r(n_1+n_2)\beta\log(n_2)}{3m}}$$

with probability at least $1 - 2n_2^{2-2\beta}$ provided that $m > \frac{16}{3}\mu_0 r(n_1+n_2)\beta\log(n_2)$.

Proof Decompose any matrix $\boldsymbol{Z}$ as $\boldsymbol{Z} = \sum_{ab}\langle \boldsymbol{Z}, \boldsymbol{e}_a\boldsymbol{e}_b^*\rangle \boldsymbol{e}_a\boldsymbol{e}_b^*$ so that

$$\mathcal{P}_T(\boldsymbol{Z}) = \sum_{ab}\langle \mathcal{P}_T(\boldsymbol{Z}), \boldsymbol{e}_a\boldsymbol{e}_b^*\rangle \boldsymbol{e}_a\boldsymbol{e}_b^* = \sum_{ab}\langle \boldsymbol{Z}, \mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*)\rangle \boldsymbol{e}_a\boldsymbol{e}_b^*. \tag{5}$$

For $k = 1,\ldots,m$ sample (a_k, b_k) from $\{1,\ldots,n_1\}\times\{1,\ldots,n_2\}$ uniformly with replacement. Then $\mathcal{R}_\Omega \mathcal{P}_T(\boldsymbol{Z}) = \sum_{k=1}^m \langle \boldsymbol{Z}, \mathcal{P}_T(\boldsymbol{e}_{a_k}\boldsymbol{e}_{b_k}^*)\rangle \boldsymbol{e}_{a_k}\boldsymbol{e}_{b_k}^*$ which gives

$$(\mathcal{P}_T \mathcal{R}_\Omega \mathcal{P}_T)(\boldsymbol{Z}) = \sum_{k=1}^m \langle \boldsymbol{Z}, \mathcal{P}_T(\boldsymbol{e}_{a_k}\boldsymbol{e}_{b_k}^*)\rangle\, \mathcal{P}_T(\boldsymbol{e}_{a_k}\boldsymbol{e}_{b_k}^*).$$

Now the fact that the operator $\mathcal{P}_T\mathcal{R}_\Omega\mathcal{P}_T$ does not deviate from its expected value

$$\mathbb{E}(\mathcal{P}_T\mathcal{R}_\Omega\mathcal{P}_T) = \mathcal{P}_T(\mathbb{E}\,\mathcal{R}_\Omega)\mathcal{P}_T = \mathcal{P}_T(\frac{m}{n_1n_2}I)\mathcal{P}_T = \frac{m}{n_1n_2}\mathcal{P}_T$$

in the spectral norm can be proven using the Noncommutative Bernstein Inequality.

To proceed, define the operator $\mathcal{T}_{ab}$ which maps $\boldsymbol{Z}$ to $\langle \mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*), \boldsymbol{Z}\rangle \mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*)$. This operator is rank one, has operator norm $\|\mathcal{T}_{ab}\| = \|\mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*)\|_F^2$, and we have $\mathcal{P}_T = \sum_{a,b}\mathcal{T}_{ab}$ by (5). Hence, for $k=1,\ldots,m$, $\mathbb{E}[\mathcal{T}_{a_kb_k}] = \frac{1}{n_1n_2}\mathcal{P}_T$.

Observe that if $\boldsymbol{A}$ and $\boldsymbol{B}$ are positive semidefinite, we have $\|\boldsymbol{A}-\boldsymbol{B}\| \le \max\{\|\boldsymbol{A}\|,\|\boldsymbol{B}\|\}$. Using this fact, we can compute the bound

$$\|\mathcal{T}_{a_kb_k} - \tfrac{1}{n_1n_2}\mathcal{P}_T\| \le \max\{\|\mathcal{P}_T(\boldsymbol{e}_{a_k}\boldsymbol{e}_{b_k}^*)\|_F^2, \tfrac{1}{n_1n_2}\} \le \mu_0 r\frac{n_1+n_2}{n_1n_2},$$

where the final inequality follows from (4). We also have

$$\begin{aligned}\|\mathbb{E}[(\mathcal{T}_{a_kb_k} - \tfrac{1}{n_1n_2}\mathcal{P}_T)^2]\| &= \|\mathbb{E}[\|\mathcal{P}_T(\boldsymbol{e}_{a_k}\boldsymbol{e}_{b_k}^*)\|_F^2\mathcal{T}_{a_kb_k}] - \frac{1}{n_1^2n_2^2}\mathcal{P}_T]\| \\ &\le \max\{\|\mathbb{E}[\|\mathcal{P}_T(\boldsymbol{e}_{a_k}\boldsymbol{e}_{b_k}^*)\|_F^2\mathcal{T}_{a_kb_k}]\|, \frac{1}{n_1^2n_2^2}\} \\ &\le \max\{\|\mathbb{E}[\mathcal{T}_{a_kb_k}]\|\mu_0 r\frac{n_1+n_2}{n_1n_2}, \frac{1}{n_1^2n_2^2}\} \le \mu_0 r\frac{n_1+n_2}{n_1^2n_2^2}.\end{aligned}$$

The theorem now follows by applying the Noncommutative Bernstein Inequality. ∎

The next theorem is an analog of Theorem 6.3 in Candès and Recht (2009) or Lemma 3.2 in Keshavan et al. (2009). This theorem asserts that for a fixed matrix, if one sets all of the entries not in Ω to zero it remains close to a multiple of the original matrix in the operator norm.

Theorem 7 *Suppose Ω is a set of entries of size m sampled independently and uniformly with replacement and let $\boldsymbol{Z}$ be a fixed $n_1\times n_2$ matrix. Assume without loss of generality that $n_1 \le n_2$, Then for all $\beta > 1$,*

$$\left\|\left(\frac{n_1n_2}{m}\mathcal{R}_\Omega - I\right)(\boldsymbol{Z})\right\| \le \sqrt{\frac{8\beta n_1n_2^2\log(n_1+n_2)}{3m}}\|\boldsymbol{Z}\|_\infty$$

with probability at least $1-(n_1+n_2)^{1-\beta}$ provided that $m > 6\beta n_1\log(n_1+n_2)$.

Proof First observe that the operator norm can be upper bounded by a multiple of the matrix infinity norm

$$\begin{aligned}\|\boldsymbol{Z}\| = \sup_{\substack{\|\boldsymbol{x}\|=1\\ \|\boldsymbol{y}\|=1}} \sum_{a,b} Z_{ab}y_ax_b &\le \left(\sum_{a,b} Z_{ab}^2y_a^2\right)^{1/2}\left(\sum_{a,b}x_b^2\right)^{1/2} \\ &\le \sqrt{n_2}\max_a\left(\sum_b Z_{ab}^2\right)^{1/2} \\ &\le \sqrt{n_1n_2}\|\boldsymbol{Z}\|_\infty.\end{aligned}$$

Note that $\frac{n_1 n_2}{m}\mathcal{R}_\Omega(\boldsymbol{Z}) - \boldsymbol{Z} = \frac{1}{m}\sum_{k=1}^m n_1 n_2 Z_{a_k b_k}\boldsymbol{e}_{a_k}\boldsymbol{e}_{b_k}^* - \boldsymbol{Z}$. This is a sum of zero-mean random matrices, and $\|n_1 n_2 Z_{a_k b_k}\boldsymbol{e}_{a_k}\boldsymbol{e}_{b_k}^* - \boldsymbol{Z}\| \leq \|n_1 n_2 Z_{a_k b_k}\boldsymbol{e}_{a_k}\boldsymbol{e}_{b_k}^*\| + \|\boldsymbol{Z}\| < \frac{3}{2}n_1 n_2\|\boldsymbol{Z}\|_\infty$ for $n_1 \geq 2$. We also have

$$\begin{aligned}\|\mathbb{E}[(n_1 n_2 Z_{a_k b_k}\boldsymbol{e}_{a_k}\boldsymbol{e}_{b_k}^* - \boldsymbol{Z})^*(n_1 n_2 Z_{a_k b_k}\boldsymbol{e}_{a_k}\boldsymbol{e}_{b_k}^* - \boldsymbol{Z})]\| & \\ &= \left\| n_1 n_2 \sum_{c,d} Z_{cd}^2 \boldsymbol{e}_d \boldsymbol{e}_d^* - \boldsymbol{Z}^*\boldsymbol{Z}\right\| \\ &\leq \max\left\{ \left\| n_1 n_2 \sum_{c,d} Z_{cd}^2 \boldsymbol{e}_d \boldsymbol{e}_d^*\right\|, \|\boldsymbol{Z}^*\boldsymbol{Z}\| \right\} \\ &\leq n_1 n_2^2 \|\boldsymbol{Z}\|_\infty^2\end{aligned}$$

where we again use the fact that $\|\boldsymbol{A}-\boldsymbol{B}\| \leq \max\{\|\boldsymbol{A}\|, \|\boldsymbol{B}\|\}$ for positive semidefinite $\boldsymbol{A}$ and $\boldsymbol{B}$. A similar calculation holds for $(n_1 n_2 Z_{a_k b_k}\boldsymbol{e}_{a_k}\boldsymbol{e}_{b_k}^* - \boldsymbol{Z})(n_1 n_2 Z_{a_k b_k}\boldsymbol{e}_{a_k}\boldsymbol{e}_{b_k}^* - \boldsymbol{Z})^*$. The theorem now follows by the Noncommutative Bernstein Inequality. ■

Finally, the following Lemma is required to prove Theorem 2. Succinctly, it says that for a fixed matrix in T, the operator $\mathcal{P}_T\mathcal{R}_\Omega$ does not increase the matrix infinity norm.

Lemma 8 *Suppose Ω is a set of entries of size m sampled independently and uniformly with replacement and let $\boldsymbol{Z} \in T$ be a fixed $n_1 \times n_2$ matrix. Assume without loss of generality that $n_1 \leq n_2$. Then for all $\beta > 2$,*

$$\left\| \frac{n_1 n_2}{m}\mathcal{P}_T\mathcal{R}_\Omega(\boldsymbol{Z}) - \boldsymbol{Z}\right\|_\infty \leq \sqrt{\frac{8\beta\mu_0 r(n_1+n_2)\log n_2}{3m}}\|\boldsymbol{Z}\|_\infty$$

with probability at least $1-2n_2^{2-\beta}$ provided that $m > \frac{8}{3}\beta\mu_0 r(n_1+n_2)\log n_2$.

Proof This lemma can be proven using the standard Bernstein Inequality. For each matrix index (c,d), sample (a,b) uniformly at random to define the random variable $\xi_{cd} = \langle \boldsymbol{e}_c\boldsymbol{e}_d^*, n_1 n_2\langle \boldsymbol{e}_a\boldsymbol{e}_b^*, \boldsymbol{Z}\rangle \mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*) - \boldsymbol{Z}\rangle$. We have $\mathbb{E}[\xi_{cd}] = 0$, $|\xi_{cd}| \leq \mu_0 r(n_1+n_2)\|\boldsymbol{Z}\|_\infty$, and

$$\begin{aligned}\mathbb{E}[\xi_{cd}^2] &= \frac{1}{n_1 n_2}\sum_{a,b}\langle \boldsymbol{e}_c\boldsymbol{e}_d^*, n_1 n_2\langle \boldsymbol{e}_a\boldsymbol{e}_b^*, \boldsymbol{Z}\rangle \mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*) - \boldsymbol{Z}\rangle^2 \\ &= n_1 n_2 \sum_{a,b}\langle \mathcal{P}_T(\boldsymbol{e}_c\boldsymbol{e}_d^*), \boldsymbol{e}_a\boldsymbol{e}_b^*\rangle^2 \langle \boldsymbol{e}_a\boldsymbol{e}_b^*, \boldsymbol{Z}\rangle^2 - Z_{cd}^2 \\ &\leq n_1 n_2 \|\mathcal{P}_T(\boldsymbol{e}_c\boldsymbol{e}_d^*)\|_F^2 \|\boldsymbol{Z}\|_\infty^2 \leq \mu_0 r(n_1+n_2)\|\boldsymbol{Z}\|_\infty^2 .\end{aligned}$$

Since the (c,d) entry of $\frac{n_1 n_2}{m}\mathcal{P}_T\mathcal{R}_\Omega(\boldsymbol{Z}) - \boldsymbol{Z}$ is identically distributed to $\frac{1}{m}\sum_{k=1}^m \xi_{cd}^{(k)}$, where $\xi_{cd}^{(k)}$ are i.i.d. copies of ξ_{cd}, we have by Bernstein's Inequality and the union bound:

$$\Pr\left[\left\| \frac{n_1 n_2}{m}\mathcal{P}_T\mathcal{R}_\Omega(\boldsymbol{Z}) - \boldsymbol{Z}\right\|_\infty > \sqrt{\frac{8\beta\mu_0 r(n_1+n_2)\log(n_2)}{3m}}\|\boldsymbol{Z}\|_\infty\right] \leq 2n_2^{2-\beta} .$$

■

4. Proof of Theorem 2

The proof follows the program developed in Gross et al. (2010) which itself adapted the strategy proposed in Candès and Recht (2009). The main idea is to approximate a dual feasible solution of (2) which certifies that $\boldsymbol{M}$ is the unique minimum nuclear norm solution. In Candès and Recht (2009) such a certificate was constructed via an infinite series using a construction developed in the compressed sensing literature (See, for example Candés et al., 2006; Fuchs, 2004). The terms in this series were then analyzed individually using the decoupling inequalities of de la Peña and Montgomery-Smith (1995). Truncating the infinite series after 4 terms gave their result. In Candès and Tao (2009), the authors bounded the contribution of $O(\log(n_2))$ terms in this series using intensive combinatorial analysis of each term. The insight in Gross et al. (2010) was that, when sampling observations with replacement, a dual feasible solution could be closely approximated by a modified series where each term involved the product of independent random variables. This change in the sampling model allows one to avoid decoupling inequalities and gives rise to the dramatic simplification here.

To proceed, recall again that by Proposition 3 it suffices to consider the scenario when the entries are sampled independently and uniformly with replacement. I will first develop the main argument of the proof assuming many conditions hold with high probability. The proof is completed by subsequently bounding probability that all of these events hold. Suppose that

$$\frac{n_1 n_2}{m}\left\|\mathcal{P}_T\mathcal{R}_\Omega\mathcal{P}_T - \frac{m}{n_1 n_2}\mathcal{P}_T\right\| \le \frac{1}{2}, \qquad \|\mathcal{R}_\Omega\| \le \tfrac{8}{3}\beta^{1/2}\log(n_2)\,. \tag{6}$$

Also suppose there exists a $\boldsymbol{Y}$ in the range of $\mathcal{R}_\Omega$ such that

$$\|\mathcal{P}_T(\boldsymbol{Y}) - \boldsymbol{U}\boldsymbol{V}^*\|_F \le \sqrt{\frac{r}{2n_2}}, \qquad \|\mathcal{P}_{T^\perp}(\boldsymbol{Y})\| < \frac{1}{2}\,. \tag{7}$$

If (6) holds, then for any $\boldsymbol{Z} \in \ker\mathcal{R}_\Omega$, $\mathcal{P}_T(\boldsymbol{Z})$ cannot be too large. Indeed, we have

$$0 = \|\mathcal{R}_\Omega(\boldsymbol{Z})\|_F \ge \|\mathcal{R}_\Omega\mathcal{P}_T(\boldsymbol{Z})\|_F - \|\mathcal{R}_\Omega\mathcal{P}_{T^\perp}(\boldsymbol{Z})\|_F\,.$$

Now observe that

$$\|\mathcal{R}_\Omega\mathcal{P}_T(\boldsymbol{Z})\|_F^2 = \langle \boldsymbol{Z}, \mathcal{P}_T\mathcal{R}_\Omega^2\mathcal{P}_T(\boldsymbol{Z})\rangle \ge \langle \boldsymbol{Z}, \mathcal{P}_T\mathcal{R}_\Omega\mathcal{P}_T(\boldsymbol{Z})\rangle \ge \frac{m}{2n_1n_2}\|\mathcal{P}_T(\boldsymbol{Z})\|_F^2$$

and $\|\mathcal{R}_\Omega\mathcal{P}_{T^\perp}(\boldsymbol{Z})\|_F \le \frac{8}{3}\beta^{1/2}\log(n_2)\|\mathcal{P}_{T^\perp}(\boldsymbol{Z})\|_F$. Collecting these facts gives that for any $\boldsymbol{Z} \in \ker\mathcal{R}_\Omega$,

$$\|\mathcal{P}_{T^\perp}(\boldsymbol{Z})\|_F \ge \sqrt{\frac{9m}{128\beta n_1 n_2\log^2(n_2)}}\|\mathcal{P}_T(\boldsymbol{Z})\|_F > \sqrt{\frac{2r}{n_2}}\|\mathcal{P}_T(\boldsymbol{Z})\|_F\,.$$

Now recall that $\|\boldsymbol{A}\|_* = \sup_{\|B\|\le 1}\langle \boldsymbol{A}, \boldsymbol{B}\rangle$. For $\boldsymbol{Z} \in \ker\mathcal{R}_\Omega$, pick $\boldsymbol{U}_\perp$ and $\boldsymbol{V}_\perp$ such that $[\boldsymbol{U}, \boldsymbol{U}_\perp]$ and $[\boldsymbol{V}, \boldsymbol{V}_\perp]$ are unitary matrices and that $\langle \boldsymbol{U}_\perp\boldsymbol{V}_\perp^*, \mathcal{P}_{T^\perp}(\boldsymbol{Z})\rangle = \|\mathcal{P}_{T^\perp}(\boldsymbol{Z})\|_*$. Then it follows that

$$\begin{aligned}
\|\boldsymbol{M}+\boldsymbol{Z}\|_* &\ge \langle \boldsymbol{U}\boldsymbol{V}^* + \boldsymbol{U}_\perp\boldsymbol{V}_\perp^*, \boldsymbol{M}+\boldsymbol{Z}\rangle \\
&= \|\boldsymbol{M}\|_* + \langle \boldsymbol{U}\boldsymbol{V}^* + \boldsymbol{U}_\perp\boldsymbol{V}_\perp^*, \boldsymbol{Z}\rangle \\
&= \|\boldsymbol{M}\|_* + \langle \boldsymbol{U}\boldsymbol{V}^* - \mathcal{P}_T(\boldsymbol{Y}), \mathcal{P}_T(\boldsymbol{Z})\rangle + \langle \boldsymbol{U}_\perp\boldsymbol{V}_\perp^* - \mathcal{P}_{T^\perp}(\boldsymbol{Y}), \mathcal{P}_{T^\perp}(\boldsymbol{Z})\rangle \\
&> \|\boldsymbol{M}\|_* - \sqrt{\frac{r}{2n_2}}\|\mathcal{P}_T(\boldsymbol{Z})\|_F + \frac{1}{2}\|\mathcal{P}_{T^\perp}(\boldsymbol{Z})\|_* \ge \|\boldsymbol{M}\|_*\,.
\end{aligned}$$

The first inequality holds from the variational characterization of the nuclear norm. We also used the fact that $\langle \boldsymbol{Y}, \boldsymbol{Z} \rangle = 0$ for all $\boldsymbol{Z} \in \ker \mathcal{R}_\Omega$. Thus, if a $\boldsymbol{Y}$ exists obeying (7), we have that for any $\boldsymbol{X}$ obeying $\mathcal{R}_\Omega(\boldsymbol{X} - \boldsymbol{M}) = \mathbf{0}$, $\|\boldsymbol{X}\|_* > \|\boldsymbol{M}\|_*$. That is, any if $\boldsymbol{X}$ has $M_{ab} = X_{ab}$ for all $(a,b) \in \Omega$, $\boldsymbol{X}$ has strictly larger nuclear norm than $\boldsymbol{M}$, and hence $\boldsymbol{M}$ is the unique minimizer of (2). The remainder of the proof shows that such a $\boldsymbol{Y}$ exists with high probability.

To this end, partition $1,\ldots,m$ into p partitions of size q. By assumption, we may choose

$$q \geq \frac{128}{3} \max\{\mu_0, \mu_1^2\} r(n_1+n_2)\beta \log(n_1+n_2) \quad \text{and} \quad p \geq \frac{3}{4}\log(2n_2).$$

Let Ω_j denote the set of indices corresponding to the jth partition. Note that each of these partitions are independent of one another when the indices are sampled with replacement. Assume that

$$\frac{n_1 n_2}{q} \left\| \mathcal{P}_T \mathcal{R}_{\Omega_k} \mathcal{P}_T - \frac{q}{n_1 n_2} \mathcal{P}_T \right\| \leq \frac{1}{2} \tag{8}$$

for all k. Define $\boldsymbol{W}_0 = \boldsymbol{U}\boldsymbol{V}^*$ and set $\boldsymbol{Y}_k = \frac{n_1 n_2}{q} \sum_{j=1}^k \mathcal{R}_{\Omega_j}(\boldsymbol{W}_{j-1})$, $\boldsymbol{W}_k = \boldsymbol{U}\boldsymbol{V}^* - \mathcal{P}_T(\boldsymbol{Y}_k)$ for $k = 1,\ldots,p$. Then

$$\begin{aligned} \|\boldsymbol{W}_k\|_F &= \left\| \boldsymbol{W}_{k-1} - \frac{n_1 n_2}{q} \mathcal{P}_T \mathcal{R}_{\Omega_k}(\boldsymbol{W}_{k-1}) \right\|_F \\ &= \left\| (\mathcal{P}_T - \frac{n_1 n_2}{q} \mathcal{P}_T \mathcal{R}_{\Omega_k} \mathcal{P}_T)(\boldsymbol{W}_{k-1}) \right\|_F \\ &\leq \frac{1}{2} \|\boldsymbol{W}_{k-1}\|_F, \end{aligned}$$

and it follows that $\|\boldsymbol{W}_k\|_F \leq 2^{-k} \|\boldsymbol{W}_0\|_F = 2^{-k}\sqrt{r}$. Since $p \geq \frac{3}{4}\log(2n_2) \geq \frac{1}{2}\log_2(2n_2) = \log_2\sqrt{2n_2}$, then $\boldsymbol{Y} = \boldsymbol{Y}_p$ will satisfy the first inequality of (7). Also suppose that

$$\left\| \boldsymbol{W}_{k-1} - \frac{n_1 n_2}{q} \mathcal{P}_T \mathcal{R}_{\Omega_k}(\boldsymbol{W}_{k-1}) \right\|_\infty \leq \frac{1}{2} \|\boldsymbol{W}_{k-1}\|_\infty, \tag{9}$$

$$\left\| \left(\frac{n_1 n_2}{q} \mathcal{R}_{\Omega_j} - \mathcal{I} \right) (\boldsymbol{W}_{j-1}) \right\| \leq \sqrt{\frac{8 n_1 n_2^2 \beta \log(n_1+n_2)}{3q}} \|\boldsymbol{W}_{j-1}\|_\infty \tag{10}$$

for $k = 1,\ldots,p$.

To see that $\|\mathcal{P}_{T^\perp}(\boldsymbol{Y}_p)\| \leq \frac{1}{2}$ when (9) and (10) hold, observe $\|\boldsymbol{W}_k\|_\infty \leq 2^{-k}\|\boldsymbol{U}\boldsymbol{V}^*\|_\infty$, and it follows that

$$\begin{aligned}
\|\mathcal{P}_{T^\perp}\boldsymbol{Y}_p\| &\leq \sum_{j=1}^{p} \|\tfrac{n_1 n_2}{q}\mathcal{P}_{T^\perp}\mathcal{R}_{\Omega_j}\boldsymbol{W}_{j-1}\| \\
&= \sum_{j=1}^{p} \|\mathcal{P}_{T^\perp}(\tfrac{n_1 n_2}{q}\mathcal{R}_{\Omega_j}\boldsymbol{W}_{j-1} - \boldsymbol{W}_{j-1})\| \\
&\leq \sum_{j=1}^{p} \|(\tfrac{n_1 n_2}{q}\mathcal{R}_{\Omega_j} - \mathcal{I})(\boldsymbol{W}_{j-1})\| \\
&\leq \sum_{j=1}^{p} \sqrt{\frac{8 n_1 n_2^2 \beta \log(n_1+n_2)}{3q}}\|\boldsymbol{W}_{j-1}\|_\infty \\
&= 2\sum_{j=1}^{p} 2^{-j}\sqrt{\frac{8 n_1 n_2^2 \beta \log(n_1+n_2)}{3q}}\|\boldsymbol{U}\boldsymbol{V}^*\|_\infty < \sqrt{\frac{32\mu_1^2 r n_2 \beta \log(n_1+n_2)}{3q}} < 1/2
\end{aligned}$$

since $q > \frac{128}{3}\mu_1^2 r n_2 \beta \log(n_1+n_2)$. The first inequality follows from the triangle inequality. The second line follows because $\boldsymbol{W}_{j-1} \in T$ for all j. The third line follows because, for any $\boldsymbol{Z}$,

$$\|\mathcal{P}_{T^\perp}(\boldsymbol{Z})\| = \|(\boldsymbol{I}_{n_1} - \boldsymbol{P}_U)\boldsymbol{Z}(\boldsymbol{I}_{n_2} - \boldsymbol{P}_V)\| \leq \|\boldsymbol{Z}\|\,.$$

The fourth line applies (10). The next line follows from (9). The final line follows from the assumption **A1**.

All that remains is to bound the probability that all of the invoked events hold. With m satisfying the bound in the main theorem statement, the first inequality in (6) fails to hold with probability at most $2n_2^{2-2\beta}$ by Theorem 6, and the second inequality fails to hold with probability at most $n_2^{2-2\beta^{1/2}}$ by Proposition 5. For all k, (8) fails to hold with probability at most $2n_2^{2-2\beta}$, (9) fails to hold with probability at most $2n_2^{2-2\beta}$, and (10) fails to hold with probability at most $(n_1+n_2)^{1-2\beta}$. Summing these all together, all of the events hold with probability at least

$$1 - 6\log(n_2)(n_1+n_2)^{2-2\beta} - n_2^{2-2\beta^{1/2}}$$

by the union bound. This completes the proof.

5. Discussion and Conclusions

The results proven here are nearly optimal, but small improvements can possibly be made. The numerical constant 32 in the statement of the theorem may be reducible by more clever bookkeeping, and it may be possible to derive a linear dependence on the logarithm of the matrix dimensions. But further reduction is not possible because of the necessary conditions provided by Candès and Tao. One minor improvement that could be made would be to remove the assumption **A1**. For instance, while μ_1 is known to be small in most of the models of low-rank matrices that have been analyzed, no one has shown that an assumption of the form **A1** is necessary for completion. Nonetheless, all prior results on matrix completion have imposed an assumption like **A1** (i.e., Candès and Recht, 2009; Candès and Tao, 2009; Keshavan et al., 2009), and it would be interesting to see if it can be removed as a requirement, or if it is somehow necessary.

In many matrix completion scenarios of interest in machine learning, the provided entries are corrupted by noise. While Theorem 2 only addresses the noise-free case, we can immediately extend our results to the noisy case. Specifically, suppose we observe $X_{ij} = M_{ij} + \nu_{ij}$ on the set Ω and we are guaranteed that

$$\sum_{(i,j)\in\Omega} \nu_{ij}^2 \leq \delta^2\,.$$

Then if we solve the quadratically constrained problem

$$\begin{array}{ll} \text{minimize} & \|\boldsymbol{X}\|_* \\ \text{subject to} & \sum_{(i,j)\in\Omega}(X_{ij} - M_{ij})^2 \leq \delta^2\,. \end{array} \tag{11}$$

we will have that any optimal solution, $\hat{\boldsymbol{M}}$ of (11) satisfies

$$\|\hat{\boldsymbol{M}} - \boldsymbol{M}\|_F \leq \left(2 + \sqrt{\frac{48 n_1^2 n_2}{m}}\right)\delta\,.$$

This claim follows directly from the argument of Candès and Plan (2009). Indeed, the only necessary requirements for such stable recovery is that (6) and (7) hold. Hence, under the sampling assumptions of Theorem 2, low-rank matrices can be approximated from noisy data by solving a quadratically constrained nuclear norm problem.

We conclude by noting that much of the simplicity of the argument presented here arises from an application of new large deviation inequalities for matrices. The noncommutative versions of Chernoff and Bernstein' s Inequalities may be useful throughout machine learning and statistical signal processing, and a fruitful line of inquiry would examine how to apply these tools from quantum information to the study of classical signals and systems.

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Appendix A. Operator Chernoff Bounds

In this section, I present a proof of 4 based on foundational results needed from Quantum Information Theory. For completeness, I also provide proofs of Theorems 9 and 10 which were originally proven in Ahlswede and Winter (2002). I have made minor modifications to the original arguments, but the assertions remain the same.

To review, a symmetric matrix $\boldsymbol{A}$ is positive semidefinite if all of its eigenvalues are nonnegative. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are positive semidefinite matrices, $\boldsymbol{A} \preceq \boldsymbol{B}$ means $\boldsymbol{B} - \boldsymbol{A}$ is positive semidefinite. For square matrices $\boldsymbol{A}$, the matrix exponential will be denoted $\exp(\boldsymbol{A})$ and is given by the power series

$$\exp(\boldsymbol{A}) = \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k}{k!}\,.$$

The following theorem is a generalization of Markov's inequality originally proven in Ahlswede and Winter (2002). Unlike the original proof, the following argument closely follows the standard proof of the traditional Markov inequality and does not rely on discrete summations.

Theorem 9 (Operator Markov Inequality) *Let $\boldsymbol{X}$ be a random positive semidefinite matrix and $\boldsymbol{A}$ a fixed positive definite matrix. Then*

$$\mathbb{P}[\boldsymbol{X} \not\preceq \boldsymbol{A}] \leq \operatorname{Tr}(\mathbb{E}[\boldsymbol{X}]\boldsymbol{A}^{-1})\,.$$

Proof Note that if $\boldsymbol{X} \not\preceq \boldsymbol{A}$, then $\boldsymbol{A}^{-1/2}\boldsymbol{X}\boldsymbol{A}^{-1/2} \not\preceq \boldsymbol{I}$, and hence $\|\boldsymbol{A}^{-1/2}\boldsymbol{X}\boldsymbol{A}^{-1/2}\| > 1$. Let $I_{\boldsymbol{X}\not\preceq\boldsymbol{A}}$ denote the indicator of the event $\boldsymbol{X} \not\preceq \boldsymbol{A}$. Then $I_{\boldsymbol{X}\not\preceq\boldsymbol{A}} \leq \operatorname{Tr}(\boldsymbol{A}^{-1/2}\boldsymbol{X}\boldsymbol{A}^{-1/2})$ as the right hand side is always nonnegative, and, if the left hand side equals 1, the trace of the right hand side must exceed the norm of the right hand side which is greater than 1. Thus we have

$$\mathbb{P}[\boldsymbol{X} \not\preceq \boldsymbol{A}] = \mathbb{E}[I_{\boldsymbol{X}\not\preceq\boldsymbol{A}}] \leq \mathbb{E}[\operatorname{Tr}(\boldsymbol{A}^{-1/2}\boldsymbol{X}\boldsymbol{A}^{-1/2})] = \operatorname{Tr}(\mathbb{E}[\boldsymbol{X}]\boldsymbol{A}^{-1})\,.$$

where the last equality follows from the linearity and cyclic properties of the trace. ■

Next I will derive a noncommutative version of the Chernoff bound. This was also proven in Ahlswede and Winter (2002) for i.i.d. matrices. The version stated here is more general in that the random matrices need not be identically distributed, but the proof is essentially the same.

Theorem 10 (Noncommutative Chernoff Bound) *Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be independent symmetric random matrices in $\mathbb{R}^{d\times d}$. Let $\boldsymbol{A}$ be an arbitrary symmetric matrix. Then for any invertible $d \times d$ matrix $\boldsymbol{T}$*

$$\mathbb{P}\left[\sum_{k=1}^{n} \boldsymbol{X}_k \not\preceq n\boldsymbol{A}\right] \leq d \prod_{k=1}^{n} \|\mathbb{E}[\exp(\boldsymbol{T}\boldsymbol{X}_{\boldsymbol{k}}\boldsymbol{T}^* - \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^*)]\|\,.$$

Proof The proof relies on an estimate of Golden (1965) and Thompson (1965) which is stated here without proof.

Lemma 11 (Golden-Thompson inequality) *For any symmetric matrices $\boldsymbol{A}$ and $\boldsymbol{B}$,*

$$\operatorname{Tr}(\exp(\boldsymbol{A}+\boldsymbol{B})) \leq \operatorname{Tr}((\exp\boldsymbol{A})(\exp\boldsymbol{B}))\,.$$

Much like the proof of the standard Chernoff bound, the theorem now follows from a long chain of inequalities.

$$\begin{aligned}
\mathbb{P}\left[\sum_{k=1}^{n} \boldsymbol{X}_k \not\preceq n\boldsymbol{A}\right] &= \mathbb{P}\left[\sum_{k=1}^{n} (\boldsymbol{X}_k - \boldsymbol{A}) \not\preceq 0\right] \\
&= \mathbb{P}\left[\sum_{k=1}^{n} \boldsymbol{T}(\boldsymbol{X}_k - \boldsymbol{A})\boldsymbol{T}^* \not\preceq 0\right] \\
&= \mathbb{P}\left[\exp\left(\sum_{k=1}^{n} \boldsymbol{T}(\boldsymbol{X}_k - \boldsymbol{A})\boldsymbol{T}^*\right) \not\preceq \boldsymbol{I}_d\right] \\
&\leq \operatorname{Tr}\left(\mathbb{E}\left[\exp\left(\sum_{k=1}^{n} \boldsymbol{T}(\boldsymbol{X}_k - \boldsymbol{A})\boldsymbol{T}^*\right)\right]\right) \\
&= \mathbb{E}\left[\operatorname{Tr}\left(\exp\left(\sum_{k=1}^{n} \boldsymbol{T}(\boldsymbol{X}_k - \boldsymbol{A})\boldsymbol{T}^*\right)\right)\right] \\
&\leq \mathbb{E}\left[\operatorname{Tr}\left(\exp\left(\sum_{k=1}^{n-1} \boldsymbol{T}(\boldsymbol{X}_k - \boldsymbol{A})\boldsymbol{T}^*\right)\exp\left(\boldsymbol{T}(\boldsymbol{X}_n - \boldsymbol{A})\boldsymbol{T}^*\right)\right)\right] \\
&\leq \mathbb{E}_{1,\dots,n-1}\left[\operatorname{Tr}\left(\exp\left(\sum_{k=1}^{n-1} \boldsymbol{T}(\boldsymbol{X}_k - \boldsymbol{A})\boldsymbol{T}^*\right)\mathbb{E}[\exp\left(\boldsymbol{T}(\boldsymbol{X}_n - \boldsymbol{A})\boldsymbol{T}^*\right)]\right)\right] \\
&\leq \|\mathbb{E}[\exp\left(\boldsymbol{T}(\boldsymbol{X}_n - \boldsymbol{A})\boldsymbol{T}^*\right)]\|\,\mathbb{E}_{1,\dots,n-1}\left[\operatorname{Tr}\left(\exp\left(\sum_{k=1}^{n-1} \boldsymbol{T}(\boldsymbol{X}_k - \boldsymbol{A})\boldsymbol{T}^*\right)\right)\right] \\
&\leq \prod_{k=2}^{n} \|\mathbb{E}[\exp\left(\boldsymbol{T}(\boldsymbol{X}_k - \boldsymbol{A})\boldsymbol{T}^*\right)]\|\,\mathbb{E}\left[\operatorname{Tr}\left(\exp\left(\boldsymbol{T}(\boldsymbol{X}_1 - \boldsymbol{A})\boldsymbol{T}^*\right)\right)\right] \\
&\leq d\prod_{k=1}^{n} \|\mathbb{E}[\exp\left(\boldsymbol{T}(\boldsymbol{X}_k - \boldsymbol{A})\boldsymbol{T}^*\right)]\|\,.
\end{aligned}$$

Here, the first three lines follow from standard properties of the semidefinite ordering. The fourth line invokes the Operator Markov Inequality. The sixth line follows from the Golden-Thompson inequality. The seventh line follows from independence of the $\boldsymbol{X}_k$. The eighth line follows because for positive definite matrices $\operatorname{Tr}(\boldsymbol{A}\boldsymbol{B}) \leq \operatorname{Tr}(\boldsymbol{A})\|\boldsymbol{B}\|$. This is just another statement of the duality between the nuclear and operator norms. The ninth line iteratively repeats the previous two steps. The final line follows because for a positive definite matrix $\boldsymbol{A}$, $\operatorname{Tr}(\boldsymbol{A})$ is the sum of the eigenvalues of $\boldsymbol{A}$, and all of the eigenvalues are at most $\|\boldsymbol{A}\|$. ■

Let us now turn to proving the Noncommutative Bernstein Inequality presented in Section 3. Gross et al. (2010) proposed a similar inequality for symmetric i.i.d. random matrices with a slightly worse constant. The proof here is more general and follows the standard derivation of Bernstein's inequality.

Proof [of Theorem 4] Set

$$\boldsymbol{Y}_k = \begin{bmatrix} \mathbf{0} & \boldsymbol{X}_k \\ \boldsymbol{X}_k^* & \mathbf{0} \end{bmatrix}.$$

Then $\boldsymbol{Y}_k$ are symmetric random variables, and for all k

$$\|\mathbb{E}[\boldsymbol{Y}_k^2]\| = \left\|\mathbb{E}\left[\left[\begin{array}{cc} \boldsymbol{X}_k\boldsymbol{X}_k^* & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{X}_k^*\boldsymbol{X}_k \end{array}\right]\right]\right\| = \max\{\|\mathbb{E}[\boldsymbol{X}_k\boldsymbol{X}_k^*]\|, \|\mathbb{E}[\boldsymbol{X}_k^*\boldsymbol{X}_k]\|\} = \rho_k^2 .$$

Moreover, the maximum singular value of $\sum_{k=1}^L \boldsymbol{X}_k$ is equal to the maximum eigenvalue of $\sum_{k=1}^L \boldsymbol{Y}_k$. By Theorem 10, we have for all $\lambda > 0$

$$\mathbb{P}\left[\left\|\sum_{k=1}^L \boldsymbol{X}_k\right\| > Lt\right] = \mathbb{P}\left[\sum_{k=1}^L \boldsymbol{Y}_k \not\preceq Lt\boldsymbol{I}\right] \leq (d_1+d_2)\exp(-L\lambda t)\prod_{k=1}^L \|\mathbb{E}[\exp(\lambda \boldsymbol{Y}_k)]\| .$$

For each k, let $\boldsymbol{Y}_k = \boldsymbol{U}_k\boldsymbol{\Lambda}_k\boldsymbol{U}_k^*$ be an eigenvalue decomposition, where $\boldsymbol{\Lambda}_k$ is the diagonal matrix of the eigenvalues of $\boldsymbol{Y}_k$. In turn, it follows that for $s > 0$

$$-M^s\boldsymbol{Y}_k^2 \preceq -\boldsymbol{U}_k M^s \boldsymbol{\Lambda}_k^2 \boldsymbol{U}_k^* \preceq \boldsymbol{U}_k\boldsymbol{\Lambda}_k^{2+s}\boldsymbol{U}_k^* = \boldsymbol{Y}_k^{2+s} \preceq \boldsymbol{U}_k M^s \boldsymbol{\Lambda}_k^2 \boldsymbol{U}_k^* \preceq M^s \boldsymbol{Y}_k^2 ,$$

which then implies

$$\|\mathbb{E}[\boldsymbol{Y}_k^{s+2}]\| \leq M^s \|\mathbb{E}[\boldsymbol{Y}_k^2]\| . \tag{12}$$

For fixed k, we have

$$\begin{aligned}
\|\mathbb{E}[\exp(\lambda \boldsymbol{Y}_k)]\| &\leq \|\boldsymbol{I}\| + \sum_{j=2}^\infty \frac{\lambda^j}{j!}\|\mathbb{E}[\boldsymbol{Y}_k^j]\| \\
&\leq 1 + \sum_{j=2}^\infty \frac{\lambda^j}{j!}\|\mathbb{E}[\boldsymbol{Y}_k^2]\| M^{j-2} \\
&= 1 + \frac{\rho_k^2}{M^2}\sum_{j=2}^\infty \frac{\lambda^j}{j!}M^j = 1 + \frac{\rho_k^2}{M^2}(\exp(\lambda M) - 1 - \lambda M) \\
&\leq \exp\left(\frac{\rho_k^2}{M^2}(\exp(\lambda M) - 1 - \lambda M)\right) .
\end{aligned}$$

The first inequality follows from the triangle inequality and the fact that $\mathbb{E}[\boldsymbol{Y}_k] = 0$, the second inequality follows from (12), and the final inequality follows from the fact that $1 + x \leq \exp(x)$ for all x. Putting this together gives

$$\mathbb{P}\left[\left\|\sum_{k=1}^L \boldsymbol{X}_k\right\| > Lt\right] \leq (d_1+d_2)\exp\left(-\lambda Lt + \frac{\sum_{k=1}^L \rho_k^2}{M^2}(\exp(\lambda M) - 1 - \lambda M)\right) .$$

This final expression is now just a real number, and only has to be minimized as a function of λ. The theorem now follows by algebraic manipulation: the right hand side is minimized by setting $\lambda = \frac{1}{M}\log(1 + \frac{tLM}{\sum_{k=1}^L \rho_k^2})$, then basic approximations can be employed to complete the argument (see, for example, Panchenko, 2007, Lectures 4 and 5). ∎

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