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# Linear Bandits on Uniformly Convex Sets

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## Abstract

Linear bandit algorithms yield $\tilde{\mathcal{O}}(n\sqrt{T})$ pseudo-regret bounds on compact convex action sets $\mathcal{K} \subset \mathbb{R}^n$ and two types of structural assumptions lead to better pseudo-regret bounds. When $\mathcal{K}$ is the simplex or an $\ell_p$ ball with $p \in ]1, 2]$, there exist bandits algorithms with $\tilde{\mathcal{O}}(\sqrt{nT})$ pseudo-regret bounds. Here, we derive bandit algorithms for some strongly convex sets beyond $\ell_p$ balls that enjoy pseudo-regret bounds of $\tilde{\mathcal{O}}(\sqrt{nT})$. This result provides new elements for the open question in (Bubeck and Cesa-Bianchi, 2012, §5.5.). When the action set is $q$-uniformly convex but not necessarily strongly convex ($q > 2$), we obtain pseudo-regret bounds $\tilde{\mathcal{O}}(n^{1/q}T^{1/p})$ with $p$ s.t. $1/p + 1/q = 1$. These pseudo-regret bounds are competitive with the general $\tilde{\mathcal{O}}(n\sqrt{T})$ for a time horizon range that depends on the degree $q > 2$ of the set's uniform convexity and the dimension $n$ of the problem.

**Keywords:** Linear Bandits, Uniformly Convex Sets, Strongly Convex Sets, Pseudo-Regret

## 1. Introduction

We consider online linear learning with partial information, a.k.a. the *linear bandit problem*. At each round $t \leq T$, the *player* (the bandit algorithm) chooses $a_t \in \mathcal{K} \subset \mathbb{R}^n$ and an *adversary* simultaneously decides on a loss vector $c_t \in \mathbb{R}^n$ (the loss function is linear). The player then observes $\langle c_t; a_t \rangle$, the loss function evaluated at the player's action $a_t$, but does not have access to the entire loss function, *i.e.*, the player does not know $c_t$. The goal of the player is to minimize its cumulative loss $\sum_{t=1}^{T} \langle c_t; a_t \rangle$. The *regret* $R_t$ compares this cumulative loss against the cumulative loss of the best single action in hindsight, *i.e.*,

$$R_T(\mathcal{K}) \triangleq \sum_{t=1}^{T} \langle c_t; a_t \rangle - \min_{a \in \mathcal{K}} \sum_{t=1}^{T} \langle c_t; a \rangle. \qquad \text{(Regret)}$$

Bandit algorithms use internal randomization to obtain sub-linear regret upper bounds. There exist several notions of regret to monitor the performance of bandit algorithms. The expected regret or an upper bound on (Regret) with high-probability are the most meaningful, yet challenging to obtain. Hence, the weaker notion of *pseudo-regret* is often considered as a good proxy for measuring the bandit performance (Bubeck and Cesa-Bianchi, 2012). It serves as a motivation to design new bandit algorithms. Let us write $\mathbb{E}$ the expectation w.r.t. the randomness of the bandit action only, we have

$$\bar{R}_T(\mathcal{K}) \triangleq \mathbb{E} \sum_{i=1}^{T} \langle c_t; a_t \rangle - \min_{a \in \mathcal{K}} \mathbb{E} \sum_{i=1}^{T} \langle c_t; a \rangle. \qquad \text{(Pseudo-Regret)}$$

We make the *bounded scalar loss assumption*, *i.e.*, $c_t$ is such that $\langle c_t; a \rangle \leq 1$ for any $a \in \mathcal{K}$. In particular, it means that $c_t$ belongs to the *polar* $\mathcal{K}^\circ \triangleq \{d \in \mathbb{R}^n \mid \langle d; x \rangle \leq 1, \forall x \in \mathcal{K}\}$ of $\mathcal{K}$.

There exist bandit algorithms with $\tilde{O}(n\sqrt{T})$ upper bounds on the *pseudo-regret* for general compact convex sets $\mathcal{K}$ (Bubeck and Cesa-Bianchi, 2012). However, since the loss is linear, it is not possible to leverage the curvature (*e.g.*, the strong convexity) of the loss function to obtain improved pseudo-regret bounds. Instead, the bandit algorithm can only leverage the specific structure of the action set $\mathcal{K}$. To the best of our knowledge, only two structures are known to induce faster pseudo-regret bounds of $\tilde{O}(\sqrt{nT})$: when $\mathcal{K}$ is a simplex or an $\ell_p$ ball with $p \in ]1, 2]$ (Bubeck et al., 2018). In each of these cases, the analysis relies on explicit analytical formulas of the action set rather than on generic quantitative properties, *e.g.*, the strong convexity of the set.

Our goal here is to design bandit algorithms that achieve pseudo-regret of $\tilde{\mathcal{O}}(\sqrt{nT})$ (resp. $\tilde{\mathcal{O}}(n^{1/q}T^{1/p})$) when the set $\mathcal{K}$ is strongly convex (resp. $q$-uniformly convex with $q \geq 2$ and $p$ s.t. $1/p + 1/q = 1$). The uniform convexity of a set is a measure of the set upper curvature that subsumes strong convexity. For instance, the $\ell_p$ balls (or also the $p$-Schatten balls) are strongly convex (Definition 3) for $p \in ]1, 2]$ and uniformly convex (Definition 4) for $p > 1$.

**Related Work.** Linear bandit algorithms are applied in a variety of applications. We detail one of them, which was our initial research motivation. Linear Bandit algorithms are instrumental in solving minimax problems with convex-linear structure stemming from learning applications, see, *e.g.*, SVMs (Hazan et al., 2011; Clarkson et al., 2012) or Distributional Robust Optimization (Namkoong and Duchi, 2016; Curi et al., 2020). In these settings, the minimax variable's linear part is a probability distribution over the dataset of size $n$. The linear bandit algorithms provide a principled framework to adaptively sample a fraction of the dataset per iteration while ensuring the convergence to a minimax optimum. The iterations' cost of the minimax algorithm is then favorably dependent on the size $n$ of the dataset. However, the dimension dependency of the linear bandit algorithm's regret bound now appears in the minimax method's convergence rate, making it crucial to design linear bandit algorithms with favorable dimension-dependent regret bounds.

Significant focus has been dedicated to designing efficient algorithms (in the full and partial feedback setting) leveraging additional properties of the loss functions such as smoothness or strong convexity (Saha and Tewari, 2011; Hazan and Levy, 2014; Garber and Kretzu, 2020, 2021) with arguably much less attention to the corresponding structural assumptions on the action sets. By studying the effect of uniform convexity of the action set in the bandit setting, we contribute to filling this gap. Note that some works recently relied on smoothness (Levy and Krause, 2019) or uniform convexity assumptions on the set in online linear learning (Huang et al., 2016, 2017; Molinaro, 2020; Kerdreux et al., 2021a) or "online learning with a hint" (Dekel et al., 2017; Bhaskara et al., 2020a,b).

At a high level, our work shares some similarities with (d'Aspremont et al., 2018) for affine-invariant analysis of accelerated first-order methods or with (Srebro et al., 2011; Rakhlin and Sridharan, 2017) in the full-information setting. Indeed, they link regret bounds of online mirror descent algorithms with the Martingale type of the ambient space. Here, we instead rely on the uniform convexity of the action set. In general, it is a more intuitive yet stronger requirement, for an explanation see, *e.g.*, (Donahue et al., 1997). It is a stronger requirement since James (1978) provides an example of space of type 2 that is non-reflexive, and hence not uniformly convex (Pettis, 1939).

**Contribution.** Our contribution are three-fold.

1. We propose a barrier function $F_{\mathcal{K}}$ for the bandit problem with strongly convex sets (more generally uniformly convex sets), *i.e.*, for $x \in \mathrm{int}(\mathcal{K})$

$$F_{\mathcal{K}}(x) \triangleq -\ln(1 - \|x\|_{\mathcal{K}}) - \|x\|_{\mathcal{K}}, \tag{Barrier}$$

   where $\|\cdot\|_{\mathcal{K}}$ is the *gauge function* to $\mathcal{K}$. For $x \in \mathbb{R}^n$, it is defined as

$$\|x\|_{\mathcal{K}} = \inf\{\lambda > 0 \mid x \in \lambda\mathcal{K}\}. \tag{Gauge}$$

2. In Theorem 11, we provide a pseudo-regret upper bound $\tilde{\mathcal{O}}(\sqrt{nT})$ for a linear bandit algorithm on some strongly convex sets. To the best of our knowledge, this setting has never been studied except in the case of the $\ell_p$ balls with $p \in ]1, 2]$. Importantly, this drastically extends the family of actions sets, *i.e.*, besides the simplex and the $\ell_p$ balls with $p \in ]1, 2]$, with such improved dimension dependency of the pseudo-regret bound in $\mathcal{O}(\sqrt{n})$. This result provides new elements for the open question from (Bubeck and Cesa-Bianchi, 2012, §5.5.).

3. When the action set is $(\alpha, q)$-uniformly convex with $q \geq 2$, we prove in Theorem 12 a pseudo-regret bound of $\tilde{\mathcal{O}}(n^{1/q}T^{1/p})$ with $p \in ]1, 2]$ s.t. $1/p + 1/q = 1$. These pseudo-regret bounds are competitive with the general $\tilde{\mathcal{O}}(n\sqrt{T})$ for a time horizon range that depends on the degree $q$ of the set's uniform convexity and the dimension $n$ of the problem, see Remark 14. The possibility for such intermediate pseudo-regret bounds was acknowledged in Bubeck et al. (2018). However, we are not aware of any existing quantitative upper bounds on the pseudo-regret in that setting.

**Outline.** In Section 2, we introduce the structural assumptions on the action sets $\mathcal{K}$ and provide some elementary results linking these structures with important quantities in the analysis. In Section 3.1, we describe the classical Mirror Descent type algorithm for bandits and our design of barrier function for uniformly convex sets, which is a generalization of the one used in Bubeck et al. (2012) by replacing the Euclidean norm with the gauge function of the action set. In Section 3.2, we then provide the convergence rates when the action sets are strongly convex (Theorem 11) and uniformly convex (Theorem 12). In Section 3.3, we present the main technical lemmas. Finally, in Appendix A, we prove the main link between uniform convexity of the set $\mathcal{K}$ and upper bounds of the Bregman Divergence of a specific function.

**Notations.** Let $\mathbb{R}^n$ be the ambient space and $(e_i)$ its canonical basis. We write $\langle \cdot ; \cdot \rangle$ for the dot product. For a norm $\|\cdot\|$, we write $\|d\|^\star = \sup_{\|x\|\leq 1}\langle x; d\rangle$ for its *dual norm*. For a convex function $f$, we write $f^*(d) \triangleq \sup_{x\in\mathbb{R}^n}\langle x; d\rangle - f(x)$ its *Fenchel conjugate*. Let $\ell_\infty(R)$ be the infinity ball with radius $R > 0$ and $\ell_1(r)$ the norm ball of $\|x\|_1 = \sum_{i=1}^n |x_i|$ with radius $r > 0$. For an open set $\mathcal{D} \subset \mathbb{R}^n$, we write $\bar{\mathcal{D}}$ its *closure*. For a compact convex set $\mathcal{K}$, we write $\partial\mathcal{K}$ its *boundary* and $\mathrm{Int}(\mathcal{K})$ its *interior*. $N_{\mathcal{K}}(x) \triangleq \left\{d \in \mathbb{R}^n \mid \langle x - y; d\rangle \geq 0 \;\; \forall y \in \mathcal{K}\right\}$ is the *normal cone* of $\mathcal{K}$ at $x$. A compact convex set $\mathcal{K}$ is strictly convex when for any $x_1, x_2 \in \partial\mathcal{K}$ s.t. $(x_1 + x_2)/2 \in \partial\mathcal{K}$, we have $x_1 = x_2$. We consider fully-dimensional compact convex sets $\mathcal{K} \subset \mathbb{R}^n$ s.t. $\ell_1(r) \subset \mathcal{K} \subset \ell_\infty(R)$ for some $r, R > 0$ which are a priori numerical constant, in particular not depending on the dimension $n$. We will consider sets $\mathcal{K}$ centrally symmetric with respect to zero with non-empty interiors. In particular, the gauge function $\|\cdot\|_{\mathcal{K}}$ of such a set is a norm (Rockafellar, 2015). For $d \in \mathbb{R}^n$, we write $\sigma_{\mathcal{K}}(d) \triangleq \sup_{x\in\mathcal{K}}\langle x; d\rangle$ the *support function* $\sigma_{\mathcal{K}}$ of $\mathcal{K}$. We have $\|\cdot\|_{\mathcal{K}}^\star = \sigma_{\mathcal{K}}$. Besides, we write $\mathcal{K}^\circ = \{d \in \mathbb{R}^n \mid \langle d; x\rangle \leq 1, \forall x \in \mathcal{K}\}$ the *polar* of $\mathcal{K}$ and we have $\|\cdot\|_{\mathcal{K}^\circ} = \sigma_{\mathcal{K}}$. We write $X \sim \mathrm{Ber}(p)$ (resp. $\mathrm{Rademacher}(p)$) a random variable $X$ following a Bernoulli (a Rademacher), *i.e.*, with values in $\{0, 1\}$ (resp. $\{-1, 1\}$) and $\mathbb{P}(X = 1) = p$.

## 2. Preliminaries

In this section, we introduce the structural assumption on $\mathcal{K}$ we will consider. Note that we will assume set smoothness (Definition 1) simply to ensure that (Barrier) is differentiable. On the contrary, the strong convexity (Definition 3) is the structure that allows for the $\sqrt{n}$ acceleration in the pseudo-regret bounds. Then we review the link between the structure of $\mathcal{K}$ and the differentiability of the set gauge function (Gauge) which then allows us to study the properties of the proposed barrier. Finally, we link upper bounds on some Bregman distance with the strong convexity of some set in Lemma 7. This will be a key inequality in our analysis.

A convex differentiable function $f$ is $L$-smooth on $\mathcal{K}$ w.r.t. $\|\cdot\|$ if and only if for any $(x, y) \in \mathcal{K}\times\mathcal{K}$

$$f(y) \leq f(x) + \langle \nabla f(x); y - x\rangle + \frac{L}{2}\|y - x\|^2. \tag{Smoothness}$$

The Hölder smoothness of a function is a relaxation of (Smoothness). For $p \in ]1, 2]$, a convex differentiable function $f$ is $(L, p)$-Hölder smooth w.r.t. $\|\cdot\|$ if and only if for any

$(x, y) \in \mathcal{K} \times \mathcal{K}$

$$f(y) \leq f(x) + \langle \nabla f(x); y - x \rangle + \frac{L}{p}\|y - x\|^p. \qquad \text{(Hölder-Smoothness)}$$

On the other hand, a set $\mathcal{K}$ is *smooth* when there is exactly one supporting hyperplane at each point of its boundary $\partial\mathcal{K}$ (Schneider, 2014). This can be defined as follows by requiring the normal cone at each boundary element to be a half-line.

**Definition 1 (Smooth Set)** *A compact convex set $\mathcal{K}$ is smooth if and only if $|N_{\mathcal{K}}(x) \cap \partial\mathcal{K}^\circ| = 1$ for any $x \in \partial\mathcal{K}$.*

One should be cautious not to confuse the smoothness of $f$ as defined in (Smoothness) and the smoothness of $\mathcal{K}$ as defined in Definition 1. Indeed, the smoothness of the set is a much weaker notion as, for instance, it implies only the differentiability of $\sigma_{\mathcal{K}}(\cdot)$, see Lemma 5. Note that not all strongly convex set are smooth. For instance, the $\ell_p$ or the $p$-Schatten balls for $p \in ]1, 2]$ are smooth and strongly convex but the elastic-net ball is strongly convex but not smooth. Also, the smoothness and strict convexity of a set are dual properties to each other in the following sense (Köthe, 1983, §26).

**Lemma 2 (Duality Set Smoothness and Strict Sonvexity)** *Consider a compact convex set $\mathcal{K} \subset \mathbb{R}^n$ with non-empty interior. Then, $\mathcal{K}$ is strictly convex if and only if $\mathcal{K}^\circ$ is smooth.*

**Proof** Let us recall the proof for completeness. Assume $\mathcal{K}$ is strictly convex and let $d \in \partial\mathcal{K}^\circ$. Let $x_1, x_2 \in \partial\mathcal{K} \cap N_{\mathcal{K}^\circ}(d)$. By definition of the normal cone $\mathcal{N}_{\mathcal{K}^\circ}(d)$, we have $\langle d; x_i \rangle \geq \langle d'; x_i \rangle$ for any $d' \in \mathcal{K}^\circ$ and $i = 1, 2$. Hence, by definition of the support function and because $x_i \in \partial\mathcal{K}$, we have $\langle d; x_i \rangle = \sup_{d' \in \mathcal{K}^\circ} \langle d'; x_i \rangle = \sigma_{\mathcal{K}^\circ}(x_i) = \|x_i\|_{\mathcal{K}} = 1$, so that

$$1 = \langle d; (x_1 + x_2)/2 \rangle \leq \|d\|_{\mathcal{K}^\circ} \|(x_1 + x_2)/2\|_{\mathcal{K}} = \|(x_1 + x_2)/2\|_{\mathcal{K}}.$$

Hence, we conclude that $(x_1 + x_2)/2 \in \partial\mathcal{K}$. Then, by strict convexity of $\mathcal{K}$ we have $x_1 = x_2$ which concludes on the smoothness of $\mathcal{K}$ (Definition 1).

Alternatively, assume that $\mathcal{K}^\circ$ is smooth. Assume by contradiction that $\mathcal{K}$ is not strictly convex, *i.e.*, that there exist different $x_1, x_2 \in \partial\mathcal{K}$ s.t. $(x_1 + x_2)/2 \in \partial\mathcal{K}$ and let $d \in N_{\mathcal{K}}((x_1 + x_2)/2) \cap \partial\mathcal{K}^\circ$. We have that $\langle d; (x_1 + x_2)/2 \rangle = \sup_{y \in \mathcal{K}} \langle d; y \rangle = \|d\|_{\mathcal{K}^\circ} = 1$. Besides, since $\langle d; (x_1 + x_2)/2 - y \rangle \geq 0$ for any $y \in \mathcal{K}$, we obtain that $\langle d; x_1 - x_2 \rangle = 0$, hence $\langle d; x_i \rangle = 1$ for $i = 1, 2$. Then, for $y \in \mathcal{K}$, we have

$$\langle d; x_1 - y \rangle = \langle d; (x_1 - x_2)/2 \rangle + \langle d; (x_1 + x_2)/2 - y \rangle \geq 0,$$

which concludes that $d \in N_{\mathcal{K}}(x_i)$ for $i = 1, 2$. Hence, this means that $x_i \in N_{\mathcal{K}^\circ}(d) \cap \partial\mathcal{K}$ for $i = 1, 2$ and contradicts the smoothness of $\mathcal{K}^\circ$ (Definition 1). ■

**Definition 3 (Set Strong Convexity)** *Let $\mathcal{K}$ be a centrally symmetric set with non-empty interior and $\alpha > 0$. $\mathcal{K}$ is $\alpha$-strongly convex w.r.t. $\|\cdot\|_{\mathcal{K}}$ if and only if for any $x, y, z \in \mathcal{K}$ and $\gamma \in [0, 1]$ we have*

$$\big(\gamma x + (1 - \gamma) y + \frac{\alpha}{2}\gamma(1 - \gamma)\|x - y\|_{\mathcal{K}}^2 z\big) \in \mathcal{K}. \qquad \text{(Set Strong Convexity)}$$

More generally, we can define the uniform convexity of a set $\mathcal{K}$ which subsumes the strong convexity. For instance the $\ell_p$ balls with $p > 2$ are uniformly convex but not strongly convex.

**Definition 4 (Set Uniform Convexity)** *Let $\mathcal{K}$ be a centrally symmetric set with non-empty interior, $\alpha > 0$, and $q \geq 2$. $\mathcal{K}$ is $(\alpha, q)$-uniformly convex w.r.t. $\|\cdot\|_{\mathcal{K}}$ if and only if for any $x, y, z \in \mathcal{K}$ and $\gamma \in [0,1]$ we have*

$$\big(\gamma x + (1-\gamma)y + \frac{\alpha}{q}\gamma(1-\gamma)\|x-y\|_{\mathcal{K}}^q z\big) \in \mathcal{K}. \qquad \text{(Set Uniform Convexity)}$$

We now recall the geometrical condition on $\mathcal{K}$ that is equivalent to differentiability of $\mathcal{K}$ (Schneider, 2014, Corollary 1.7.3.).

**Lemma 5 (Gauge Differentiability)** *A gauge function $\|\cdot\|_{\mathcal{K}}$* (Gauge) *is differentiable at $x \in \mathbb{R}^n \setminus \{0\}$ if and only if its support set*

$$S(\mathcal{K}^\circ, x) \triangleq \{d \in \mathcal{K}^\circ \ : \ \langle d; x\rangle = \sup_{d' \in \mathcal{K}^\circ} \langle d'; x\rangle\}, \qquad \text{(Support Set)}$$

*contains a single point $d$. If this is the case, we have $\nabla\|\cdot\|_{\mathcal{K}}(x) = d$. Besides, the following assertions are true*

*(a)* $\big\|\big(\nabla\|\cdot\|_{\mathcal{K}}(x)\big)\big\|_{\mathcal{K}^\circ} = 1$*, i.e.,* $\nabla\|\cdot\|_{\mathcal{K}}(x) \in \mathcal{K}^\circ$.

*(b) For $\lambda > 0$, $\nabla\|\cdot\|_{\mathcal{K}}(\lambda x) = \nabla\|\cdot\|_{\mathcal{K}}(x)$.*

*(c) If $\mathcal{K}^\circ$ is strictly convex then $\|\cdot\|_{\mathcal{K}}$ is differentiable on $\mathbb{R}^n \setminus \{0\}$.*

**Proof** The differentiability result for $\|\cdot\|_{\mathcal{K}}$ comes from (Schneider, 2014, Corollary 1.7.3.), where we used that $\|\cdot\|_{\mathcal{K}} = \sigma_{\mathcal{K}^\circ}$. (a) follows from the fact that the supremum in (Support Set) is attained at $\partial\mathcal{K}^\circ$. For $\lambda > 0$, we have $S(K^\circ, \lambda x) = S(K^\circ, x)$ and hence (b). Now assume that $\mathcal{K}^\circ$ is strictly convex and consider $x \in \mathbb{R}^n \setminus \{0\}$. First remark as for (a) that $S(\mathcal{K}^\circ, x) \subset \partial\mathcal{K}^\circ$. Assume that $|S(\mathcal{K}^\circ, x)| \neq 1$. Then, for $d_1, d_2$ distinct in $S(\mathcal{K}^\circ, x)$, we have $[d_1, d_2] \subset S(\mathcal{K}^\circ, x) \subset \partial\mathcal{K}^\circ$ which then contradicts the strict convexity of $\mathcal{K}^\circ$. Hence $|S(\mathcal{K}^\circ, x)| = 1$ which concludes (c). ∎

**Definition 6 (Bregman Divergence)** *The Bregman divergence of $F : \mathcal{D} \to \mathbb{R}$ is defined for $(x, y) \in \bar{\mathcal{D}} \times \mathcal{D}$ by*

$$D_F(x, y) = F(x) - F(y) - \langle x - y; \nabla F(y)\rangle. \qquad \text{(Bregman Divergence)}$$

The strong-convexity assumption on $\mathcal{K}$ appears in the analysis of Algorithm 1 via an upper bound on the (Bregman Divergence) of $\frac{1}{2}\|\cdot\|_{\mathcal{K}^\circ}$. Indeed, when $\mathcal{K}$ is *strongly convex*, then $\mathcal{K}^\circ$ is strongly smooth and hence $\sigma_{\mathcal{K}}^2$ is $L$-smooth with respect to $\|\cdot\|_{\mathcal{K}}$, see (Kerdreux et al., 2021b, Theorem 4.1.) that we recall in Theorem 20 in the Appendix A. It then implies the following upper bound on its Bregman Divergence.

**Lemma 7 (Upper-bound on the Bregman Divergence of $\frac{1}{2}\|\cdot\|^2_{\mathcal{K}^\circ}$)** *Let $q \geq 2$ and $p \in ]1,2]$ s.t. $1/p + 1/q = 1$. Let $\mathcal{K}$ be a centrally symmetric set with non-empty interior. Assume $\mathcal{K}$ is $(\alpha, q)$-uniformly convex with respect to $\|\cdot\|_{\mathcal{K}}$. Then, for any $(u,v) \in \mathbb{R}^n$, we have*

$$D_{\frac{1}{2}\|\cdot\|^2_{\mathcal{K}^\circ}}(u,v) \leq 2p\big(1 + (q/(2\alpha))^{1/(q-1)}\big)\|u-v\|^p_{\mathcal{K}^\circ}. \tag{1}$$

**Proof** For a $(L,r)$-Hölder smooth function $f$ w.r.t. to $\|\cdot\|$ we immediately have $D_f(u,v) \leq \frac{L}{r}\|u-v\|^r$. Theorem 20 implies that $\frac{1}{2}\|\cdot\|^2_{\mathcal{K}^\circ}$ is $(L,p)$-Hölder Smooth on $\mathcal{K}^\circ$ w.r.t. $\|\cdot\|_{\mathcal{K}^\circ}$ where $L = 2p\big(1 + \big(\frac{q}{2\alpha}\big)^{1/(q-1)}\big)$. This concludes the proof. ■

We immediately obtain the following corollary for the strongly convex case with $p = q = 2$.

**Corollary 8 (Strongly Convex Case)** *Let $\mathcal{K}$ be a centrally symmetric set with non-empty interior. Assume $\mathcal{K}$ is $\alpha$-strongly convex with respect to $\|\cdot\|_{\mathcal{K}}$. Then for any $(u,v) \in \mathbb{R}^n$, we have*

$$D_{\frac{1}{2}\|\cdot\|^2_{\mathcal{K}^\circ}}(u,v) \leq 4\Big(\frac{\alpha+1}{\alpha}\Big)\big\|u-v\big\|^2_{\mathcal{K}^\circ}.$$

## 3. Pseudo-Regret Bounds of Linear Bandit on Strongly Convex Sets

In Section 3.1, we first present the algorithm and barrier function for linear bandits on uniformly convex sets. In Section 3.2, we then present the main pseudo-regret bounds and the proofs of the technical lemmas are relegated in Section 3.3.

### 3.1 Mirror Descent for Bandits

We propose to use a similar bandit algorithm to the one developed in (Bubeck and Cesa-Bianchi, 2012) for linear bandits over the Euclidean ball. Namely, Algorithm 1 is an instantiation of Online Stochastic Mirror Descent (OSMD) with a carefully designed barrier function $F_{\mathcal{K}} : \mathrm{Int}(\mathcal{K}) \to \mathbb{R}^+$. For any $x \in \mathrm{Int}(\mathcal{K})$ we defined in (Barrier)

$$F_{\mathcal{K}}(x) = -\ln(1 - \|x\|_{\mathcal{K}}) - \|x\|_{\mathcal{K}}.$$

Algorithm 1 keeps track of a sequence of vectors $x_t \in (1-\gamma)\mathcal{K}$ and at each iteration samples an action $a_t \in \mathcal{K}$ as described in Lines 4-8. For some $r > 0$, we assume $\ell_1(r) \subset \mathcal{K}$ so that $re_i \in \mathcal{K}$. After playing action $a_t \in \mathcal{K}$, the bandit receives the loss $\langle c_t; a_t\rangle$ associated to its action without observing the full vector $c_t \in \mathcal{K}^\circ$. In Line 10, it then proposes an unbiased estimation $\tilde{c}_t$ of $c_t$. Indeed, we have (because $\mathbb{P}(\xi_t = 0) = 1 - \|x\|_{\mathcal{K}}$)

$$\mathbb{E}_{\xi_t, i_t, \epsilon_t}(\tilde{c}_t) = \mathbb{P}(\xi_t = 0)\sum_{i=1}^n \frac{n}{r^2}\frac{1}{n}\Big[\frac{\langle re_i; c_t\rangle}{2(1-\|x\|_{\mathcal{K}})}re_i + \frac{\langle -re_i; c_t\rangle}{2(1-\|x\|_{\mathcal{K}})}(-re_i)\Big] = c_t.$$

The bandit then provides the vector $\tilde{c}_t$ to an online learning algorithm that updates the $x_t$ vector in Line 11. Importantly, because $x_t \in (1-\gamma)\mathcal{K}$ with $\gamma \in ]1,2[$ we have $\|x_t\| < 1$ so that $\nabla F_{\mathcal{K}}(x_t)$ is well defined.

**Algorithm 1:** Bandit Mirror Descent (BMD) on some Curved Sets $\mathcal{K} \subset \mathbb{R}^n$

**Input:** $\eta > 0$, $\gamma \in ]0, 1[$, $\mathcal{K}$ smooth and strictly convex s.t. $\ell_1(r) \subset \mathcal{K}$.
1 **Barrier:** $F_{\mathcal{K}}(\cdot) = -\ln(1 - \|\cdot\|_{\mathcal{K}}) - \|\cdot\|_{\mathcal{K}}$.
2 **Initialize:** $x_1 \in \text{argmin}_{x \in (1-\gamma)\mathcal{K}} F_{\mathcal{K}}(x)$.
3 **for** $t \leftarrow 1, \ldots, T$ **do**
4 Sample $\xi_t \sim \text{Ber}(\|x_t\|_{\mathcal{K}})$, $i_t \sim \text{Uniform}(n)$ and $\epsilon_t \sim \text{Rademacher}(\frac{1}{2})$.
5 **if** $\xi_t = 1$ **then**
6 $a_t \leftarrow x_t / \|x_t\|_{\mathcal{K}}$. ▷ Define bandit action.
7 **else**
8 $a_t \leftarrow r \epsilon_t e_{i_t}$.
9 **end**
10 $\tilde{c}_t \leftarrow \frac{n}{r^2}(1 - \xi_t) \frac{\langle a_t; c_t \rangle}{1 - \|x_t\|_{\mathcal{K}}} a_t$. ▷ Estimate full loss vector $c_t$.
11 $x_{t+1} \leftarrow \underset{y \in (1-\gamma)\mathcal{K}}{\text{argmin}}\ D_{F_{\mathcal{K}}}\big(y, \nabla F^*_{\mathcal{K}}(\nabla F_{\mathcal{K}}(x_t) - \eta \tilde{c}_t)\big)$. ▷ Mirror Descent step.
12 **end**
**Output:** $\frac{1}{T} \sum_{t=1}^{T} a_t$

To ensure that Line 11 of Algorithm 1 is well defined, we need to check, *e.g.*, that all $x_t$ belongs of $\text{Int}(\mathcal{K})$ (which we know is the case because $x_t \in (1-\gamma)\mathcal{K}$) or that $\nabla F_{\mathcal{K}}(x_t) - \eta \tilde{c}_T$ belongs to $\mathcal{D}^*_{\mathcal{K}}$ the domain where $F^*_{\mathcal{K}}$ is defined. In Lemma 10 below, we guarantee that Algorithm 1 is well defined. We also prove that $F_{\mathcal{K}}$ is Legendre (Definition 9) which allows us to invoke classical convergence results as in Bubeck and Cesa-Bianchi (2012).

**Definition 9 (Legendre Function)** *A continuous function $F : \bar{\mathcal{D}} \rightarrow \mathbb{R}$ is* Legendre *if and only if*

*(a) $F$ is strictly convex and admits continuous first partial derivatives on $\mathcal{D}$.*

*(b)* $\lim\limits_{x \rightarrow \bar{\mathcal{D}} \setminus \mathcal{D}} \|\nabla F(x)\| = +\infty$.

**Lemma 10 (Barrier $F_{\mathcal{K}}$ for $\mathcal{K}$)** *Consider a compact, smooth and strictly convex $\mathcal{K}$. We consider for $x \in \mathcal{D}_{\mathcal{K}} \triangleq \big\{x \in \mathbb{R}^n \mid \|x\|_{\mathcal{K}} < 1\big\}$ the following barrier function as defined in* (Barrier)

$$F_{\mathcal{K}}(x) = -\ln(1 - \|x\|_{\mathcal{K}}) - \|x\|_{\mathcal{K}}.$$

*Then $F$ is Legendre (Definition 9) with $\mathcal{D}^*_{\mathcal{K}} = \mathbb{R}^n$ and $\mathcal{K} \subset \bar{\mathcal{D}}_{\mathcal{K}}$.*

**Proof** From Lemma 15, because $\mathcal{K}$ is smooth and strictly convex, $F_{\mathcal{K}}$ (resp. $F^*_{\mathcal{K}}$) is differentiable on $\text{Int}(\mathcal{K})$ (resp. $\mathbb{R}^n$). Besides, we have $\mathcal{D}^*_{\mathcal{K}} = \mathbb{R}^n$. Finally, the strict convexity of $F$ comes from the strict convexity of $\|\cdot\|_{\mathcal{K}}$ when $\mathcal{K}$ is strictly convex. Hence $F$ is Legendre. ■

### 3.2 Main Result

Although uniform convexity subsumes strong convexity, for the sake of clarity, we first state in Theorem 11 the pseudo-regret upper bounds of Algorithm 1 when the set is strongly convex. In Theorem 12, we then extend these convergence results to the case where the action set is more generally uniformly convex.

**Theorem 11 (Linear Bandit on Strongly Convex Set)** *Consider a compact convex set $\mathcal{K}$ that is centrally symmetric with non-empty interior. Assume $\mathcal{K}$ is smooth and $\alpha$-strongly convex set w.r.t. $\|\cdot\|_{\mathcal{K}}$ and $\ell_2(r) \subset \mathcal{K} \subset \ell_\infty(R)$ for some $r, R > 0$. Consider running BMD (Algorithm 1) with the barrier function $F_{\mathcal{K}}(x) = -\ln\big(1-\|x\|_{\mathcal{K}}\big) - \|x\|_{\mathcal{K}}$, and*

$$\eta = \frac{1}{\sqrt{nT}},\ \gamma = \frac{1}{\sqrt{T}}. \tag{2}$$

*For $T \geq 4n\big(\frac{R}{r}\big)^2$ we then have*

$$\bar{R}_T \leq \sqrt{T} + \sqrt{nT}\ln(T)/2 + L\sqrt{nT} = \tilde{\mathcal{O}}(\sqrt{nT}), \tag{Pseudo-Regret Upper-Bound}$$

*where $\bar{R}_T$ is defined in* (Pseudo-Regret) *and $L = (R/r)^2(5\alpha+4)/\alpha$.*

**Proof** First note that with $T \geq 4n(R/r)^2$ and $\eta = 1/\sqrt{nT}$, we have that $\eta \leq r/(2Rn)$ which allows to invoke Lemma 17. The proof follows that of (Bubeck and Cesa-Bianchi, 2012, Theorem 5.8) but importantly leverages on our novel Lemma 17 that carefully upper bounds the terms $D_{F^*_{\mathcal{K}}}(\nabla F_{\mathcal{K}}(x_t) - \eta\tilde{c}_t, \nabla F_{\mathcal{K}}(x_t))$ for the barrier function we designed. Because $F_{\mathcal{K}}$ is Legendre and $\tilde{c}_t$ is an unbiased estimate of $c_t$, by (Bubeck and Cesa-Bianchi, 2012, Theorem 5.5) applied on $\mathcal{K}' \triangleq (1-\gamma)\mathcal{K}$, we have

$$\bar{R}_T(\mathcal{K}') \leq \frac{\sup_{x\in(1-\gamma)\mathcal{K}} F_{\mathcal{K}}(x) - F_{\mathcal{K}}(x_1)}{\eta} + \frac{1}{\eta}\sum_{t=1}^{T}\mathbb{E}\Big[D_{F^*_{\mathcal{K}}}\big(\nabla F_{\mathcal{K}}(x_t) - \eta\tilde{c}_t, \nabla F_{\mathcal{K}}(x_t)\big)\Big].$$

Also, by definition of the Pseudo-Regret, we have

$$\bar{R}_T(\mathcal{K}) = \bar{R}_T(\mathcal{K}') + \min_{a\in\mathcal{K}'}\sum_{i=1}^{T}\langle c_t; a\rangle - \min_{a\in\mathcal{K}}\sum_{i=1}^{T}\langle c_t; a\rangle.$$

Write $a^* \in \mathcal{K}$ for which $\min_{a\in\mathcal{K}}\sum_{i=1}^{T}\langle c_t; a\rangle$ is attained. We have that the $\min_{a\in\mathcal{K}'}\sum_{i=1}^{T}\langle c_t; a\rangle$ is attained at $(1-\gamma)a^*$, hence because $|\langle c_t; a^*\rangle| \leq 1$ for any $t$, we have

$$\bar{R}_T(\mathcal{K}) = \bar{R}_T(\mathcal{K}') + \sum_{i=1}^{T}\langle c_t; (1-\gamma)a^*\rangle - \sum_{i=1}^{T}\langle c_t; a^*\rangle = \bar{R}_T(\mathcal{K}') - \gamma\sum_{i=1}^{T}\langle c_t; a^*\rangle \leq \bar{R}_T(\mathcal{K}') + \gamma T.$$

By the initialization of $x_1$ in Line 2 of Algorithm 1, we have $F_{\mathcal{K}}(x_1) = 0$. Besides, by definition of $F_{\mathcal{K}}$, $\sup_{x\in\mathcal{K}} F_{\mathcal{K}}(x) \leq \ln(1/\gamma)$, so that $\sup_{x\in\mathcal{K}} F(x) - F_{\mathcal{K}}(x_1) \leq \ln(1/\gamma)$. Overall, we have

$$\bar{R}_T(\mathcal{K}) \leq \gamma T + \frac{\ln(1/\gamma)}{\eta} + \frac{1}{\eta}\sum_{t=1}^{T}\mathbb{E}\Big[D_{F^*_{\mathcal{K}}}\big(\nabla F_{\mathcal{K}}(x_t) - \eta\tilde{c}_t, \nabla F_{\mathcal{K}}(x_t)\big)\Big].$$

We have $\eta \leq r/(2Rn)$ and hence Lemma 17 implies that

$$\bar{R}_T(\mathcal{K}) \leq \gamma T + \frac{\ln(1/\gamma)}{\eta} + \eta\Big(1 + \frac{4(\alpha+1)}{\alpha}\Big)\sum_{t=1}^{T}\mathbb{E}\Big((1-\|x\|_{\mathcal{K}})\|\tilde{c}_t\|^2_{\mathcal{K}^\circ}\Big).$$

Then, let us explicit $\mathbb{E}\Big((1-\|x\|_{\mathcal{K}})\|\tilde{c}_t\|^2_{\mathcal{K}^\circ}\Big)$. Recall that $\mathcal{K} \subset \ell_\infty(R)$, so that $\ell_\infty^\circ(R) = \ell_1(1/R) \subset \mathcal{K}^\circ$ and $e_i/R \in \mathcal{K}^\circ$. Hence, we have that $\|re_i\|_{\mathcal{K}^\circ} = rR\|e_i/R\|_{\mathcal{K}^\circ} \leq rR$. We obtain

$$\begin{aligned}
\mathbb{E}\Big((1-\|x\|_{\mathcal{K}})\|\tilde{c}_t\|^2_{\mathcal{K}^\circ}\Big) &= \mathbb{P}(\xi_t = 0)\sum_{i=1}^{n}\frac{1}{n}(1-\|x_t\|_{\mathcal{K}})\frac{n^2}{r^4}\Big(\frac{\langle re_i; c_t\rangle}{1-\|x_t\|_{\mathcal{K}}}\Big)^2\|re_i\|^2_{\mathcal{K}^\circ}\\
&\leq (1-\|x_t\|_{\mathcal{K}})\sum_{i=1}^{n} nR^2\frac{c_{t,i}^2}{1-\|x_t\|_{\mathcal{K}}} = nR^2\|c_t\|_2^2.
\end{aligned}$$

We have $\ell_2(r) \subset \mathcal{K}$. This implies $\mathcal{K}^\circ \subset \ell_2(r)^\circ = \ell_2(1/r)$ so that with $c_t \in \mathcal{K}^\circ$, we have $\|c_t\|_2^2 \leq 1/r^2$. Hence

$$\bar{R}_T(\mathcal{K}) \leq \gamma T + \frac{\ln(1/\gamma)}{\eta} + \eta\Big(1+\frac{4(\alpha+1)}{\alpha}\Big)n\Big(\frac{R}{r}\Big)^2 T,$$

and we immediately obtain (Pseudo-Regret Upper-Bound) with the prescribed choice of $\eta$ and $\gamma$. ∎

**Theorem 12 (Linear Bandit on Uniformly Convex Sets)** *Let $\alpha > 0$, $q \geq 2$, and $p \in ]1, 2]$ s.t. $1/p + 1/q = 1$. Consider a compact convex set $\mathcal{K}$ that is centrally symmetric with non-empty interior. Assume $\mathcal{K}$ is smooth and $(\alpha, q)$-uniformly convex set w.r.t. $\|\cdot\|_{\mathcal{K}}$ and $\ell_q(r) \subset \mathcal{K} \subset \ell_\infty(R)$ for some $r, R > 0$. Consider running BMD (Algorithm 1) with the barrier function $F_{\mathcal{K}}(x) = -\ln\big(1-\|x\|_{\mathcal{K}}\big) - \|x\|_{\mathcal{K}}$, and*

$$\eta = 1/(n^{1/q}T^{1/p}),\ \gamma = 1/\sqrt{T}. \tag{3}$$

*Then we have for $T \geq 2^p n\big(\frac{R}{r}\big)^p$*

$$\bar{R}_T \leq \sqrt{T} + n^{1/q}T^{1/p}\ln(T)/2 + ((1/2)^{2-p} + L)\Big(\frac{R}{r}\Big)^p n^{1/q}T^{1/p} = \tilde{\mathcal{O}}(n^{1/q}T^{1/p}),$$

*where $\bar{R}_T$ is defined in* (Pseudo-Regret) *and $L = 2p(1 + (q/(2\alpha))^{1/(q-1)})$.*

**Proof** The proof is similar to Theorem 11 and hence to (Bubeck and Cesa-Bianchi, 2012, Theorem 5.8). The difference is that we now leverage Corollary 18. Note that with $T \geq 2^p n(R/r)^p$ and $\eta = n^{-1/q}T^{-1/p}$, we have $0 \leq \eta \leq 1/(2n)(r/R)$. As in the proof of Theorem 11, we have

$$\bar{R}_T(\mathcal{K}) \leq \gamma T + \frac{\ln(1/\gamma)}{\eta} + \frac{1}{\eta}\sum_{t=1}^{T}\mathbb{E}\Big[D_{F^*_{\mathcal{K}}}\big(\nabla F_{\mathcal{K}}(x_t) - \eta\tilde{c}_t, \nabla F_{\mathcal{K}}(x_t)\big)\Big].$$

Now applying Corollary 18, we have with $L = 2p(1 + (q/(2\alpha))^{1/(q-1)})$

$$D_{F^*_{\mathcal{K}}}(\nabla F_{\mathcal{K}}(x_t) - \eta\tilde{c}_t, \nabla F_{\mathcal{K}}(x_t)) \leq (1 - \|x_t\|_{\mathcal{K}})\eta^p \|\tilde{c}_t\|^p_{\mathcal{K}^\circ}((1/2)^{2-p} + L).$$

This hence implies

$$\bar{R}_T(\mathcal{K}) \leq \gamma T + \frac{\ln(1/\gamma)}{\eta} + \eta^{p-1}((1/2)^{2-p} + L)\sum_{t=1}^{T} \mathbb{E}\Big[(1 - \|x\|_{\mathcal{K}})\|\tilde{c}_t\|^p_{\mathcal{K}^\circ}\Big].$$

Let us now upper bound $\mathbb{E}\Big[(1 - \|x\|_{\mathcal{K}})\|\tilde{c}_t\|^p_{\mathcal{K}^\circ}\Big]$. Since $\mathcal{K} \subset \ell_\infty(R)$, we have $\ell_1(1/R) \subset \mathcal{K}^\circ$ and $e_i/R \in \mathcal{K}^\circ$ so that $\|re_i\|_{\mathcal{K}^\circ} \leq rR$. Hence, we have

$$\begin{aligned}
\mathbb{E}\Big((1 - \|x\|_{\mathcal{K}})\|\tilde{c}_t\|^p_{\mathcal{K}^\circ}\Big) &= \mathbb{P}(\xi_t = 0)\sum_{i=1}^{n} \frac{1}{n}(1 - \|x_t\|_{\mathcal{K}})\Big(\frac{n}{r^2}\Big)^p \Big(\frac{|\langle re_i; c_t\rangle|}{1 - \|x_t\|_{\mathcal{K}}}\Big)^p \|re_i\|^p_{\mathcal{K}^\circ} \\
&\leq (1 - \|x_t\|_{\mathcal{K}})^{2-p}\sum_{i=1}^{n} n^{p-1}R^p c^p_{t,i} \leq n^{p-1}R^p\|c_t\|^p_p.
\end{aligned}$$

Then since $\ell_q(r) \subset \mathcal{K}$, we have $\mathcal{K}^\circ \subset \ell_q(r)^\circ = \ell_p(1/r)$ so that $\|c_t\|_p \leq 1/r$ because $c_t \in \mathcal{K}^\circ$. We ultimately obtain

$$\bar{R}_T(\mathcal{K}) \leq \gamma T + \frac{\ln(1/\gamma)}{\eta} + \eta^{p-1}((1/2)^{2-p} + L)Tn^{p-1}\Big(\frac{R}{r}\Big)^p.$$

Here, we choose $\eta$ of the form $T^{-\beta}n^{-\nu}$ with $\beta$ and $\nu$ such that the terms $1/\eta$ and $\eta^{p-1}Tn^{p-1}$ exhibit the same asymptotic rate in $n$ and $T$ respectively. In particular, we choose $\eta = 1/(n^{1/q}T^{1/p})$ and obtain (with $\gamma = 1/\sqrt{T}$)

$$\bar{R}_T(\mathcal{K}) \leq \sqrt{T} + n^{1/q}T^{1/p}\ln(T)/2 + ((1/2)^{2-p} + L)\Big(\frac{R}{r}\Big)^p n^{1/q}T^{1/p}.$$

■

Instantiating the regret bound in Theorem 12 with $p = q = 2$ results in the same regret bound as in Theorem 11. Indeed, the parameters in (3) with $q = 2$ correspond to (2).

**Remark 13** *Consider two compact convex sets $\mathcal{K}_1$ and $\mathcal{K}_2$. Their* relative width *is defined as follows*

$$w(\mathcal{K}_1, \mathcal{K}_2) \triangleq \underset{x\in\mathcal{K}_1, y\in\mathcal{K}_2}{sup} \langle x; y\rangle. \qquad \text{(Relative-Width)}$$

*Note that $w(\mathcal{K}_1, \mathcal{K}_2) = sup_{x\in\mathcal{K}_1}\|x\|_{\mathcal{K}_2^\circ}$ and $\ell_q(r)^\circ = \ell_p(1/r)$, using the* (Relative-Width) *we could replace the condition $\ell_q(r) \subset \mathcal{K}$ by $w(\mathcal{K}^\circ, \ell_q(1)) \leq 1/r$.*

**Remark 14** *(Bubeck et al., 2018, Theorem 4) proves that for a sufficiently large time-horizon $T \geq n^{max\left(2; \frac{q-1}{q-2}\right)}$ the pseudo-regret of linear bandit when the action set is an $\ell_q$ ball with $q > 2$ is lower bounded by $n\sqrt{T}$. Similarly, our pseudo-regret bound $\tilde{\mathcal{O}}(n^{1/q}T^{1/p})$ is competitive w.r.t. the general $\tilde{\mathcal{O}}(n\sqrt{T})$ for some intermediate regime of the time horizon,* *i.e., when $T = \Omega(n^\alpha)$ with $\alpha \in [1, 2/(2-p)[$ and $1/p + 1/q = 1$. In particular, when $q$ converges to 2, the pseudo-regret bounds is better to the general $\tilde{\mathcal{O}}(n\sqrt{T})$ for increasingly large time horizon regimes $T$. This is consistent with the fact that the pseudo-regret bound $\tilde{\mathcal{O}}(\sqrt{nT})$ on strongly convex is competitive with a $\tilde{\mathcal{O}}(n\sqrt{T})$ for any time horizon $T \geq \Omega(n)$. However, note that the time horizon regime at which our lower bound is not anymore competitive with the general $\tilde{\mathcal{O}}(n\sqrt{T})$ differs from that of (Bubeck et al., 2018). Indeed, the lower bound of (Bubeck et al., 2018, Theorem 4) is valid with less stringent requirements on the range of the losses. Namely, we require the loss to belong to the polar of the action set, i.e., the $\ell_p$ ball. In contrast, (Bubeck et al., 2018, Theorem 4) only requires the expected loss vector to belong to the $\ell_p$ ball.*

### 3.3 Technical Lemmas

We now detail the lemmas invoked in the proofs of Theorems 11 and 12. Lemma 15 provides the expression for $\nabla F_{\mathcal{K}}$ and $\nabla F_{\mathcal{K}}^*$ and their differentiability domain. Lemma 16 is a technicality that notably explains why we constrain $\eta$ in $[0, r/(2nR)]$. Lemma 17 (resp. Corollary 18) are instrumental in upper-bounding the terms $D_{F_{\mathcal{K}}^*}(\nabla F_{\mathcal{K}}(x_t) - \eta\tilde{c}_t, \nabla F_{\mathcal{K}}(x_t))$ when the set is strongly convex (resp. uniformly convex). Technically, we build the link between the uniform convexity of the set and upper bounds on the regret in these lemmas. Although uniform convexity is a weaker assumption than strong convexity, we distinguish the cases to stress the convergence results when the action sets are strongly convex. All lemmas are self-contained and stated independently from Algorithm 1.

**Lemma 15 (Some Identities)** *Assume $\mathcal{K} \subset \mathbb{R}^n$ is strictly convex compact and smooth set. Let $x \in \mathcal{K}$ s.t. $\|x\|_{\mathcal{K}} < 1$ and $d \in \mathbb{R}^n \setminus \{0\}$. With $F_{\mathcal{K}}(x) = -\ln\big(1 - \|x\|_{\mathcal{K}}\big) - \|x\|_{\mathcal{K}}$, $F_{\mathcal{K}}$ (resp. $F_{\mathcal{K}}^*$) is differentiable on Int($\mathcal{K}$) (resp. $\mathbb{R}^n$) and we have*

$$
\begin{cases}
\nabla F_{\mathcal{K}}(x) = \dfrac{\|x\|_{\mathcal{K}}}{1 - \|x\|_{\mathcal{K}}} \nabla \| \cdot \|_{\mathcal{K}}(x) \\
F_{\mathcal{K}}^*(d) = \|d\|_{\mathcal{K}^\circ} - \ln(1 + \|d\|_{\mathcal{K}^\circ}) \\
\nabla F_{\mathcal{K}}^*(d) = \dfrac{\|d\|_{\mathcal{K}^\circ}}{1 + \|d\|_{\mathcal{K}^\circ}} \nabla \| \cdot \|_{\mathcal{K}^\circ}(d).
\end{cases}
\tag{4}
$$

**Proof** Let us first compute $F_{\mathcal{K}}^*$. We have $F_{\mathcal{K}}(d) = g \circ \|x\|_{\mathcal{K}}$ with $g(r) = -\ln(1-r) - r$ for $r \in [0, 1[$. Note that $g(0) = 0$ and $g$ is convex. Write $g^*(y) \triangleq \sup_{r \in [0,1]} yr + \ln(1-r) + r$ for $y \geq 0$. With simple analysis, we have $g^*(y) = y - \ln(1+y)$. Then, with, *e.g.*, (Schneider, 2014, 1.47), we have that $F_{\mathcal{K}}^*(d) = g^* \circ \|d\|_{\mathcal{K}^\circ} = \|d\|_{\mathcal{K}^\circ} - \ln(1 + \|d\|_{\mathcal{K}^\circ})$.

The gradient identities (4) are then immediate at points $(x, d)$ s.t. $\| \cdot \|_{\mathcal{K}}$ and $\| \cdot \|_{\mathcal{K}^\circ}$ are differentiable. From Lemma 2 since $\mathcal{K}$ is smooth, $\mathcal{K}^\circ$ is strictly convex. For $(x, d) \in \mathcal{K} \setminus \{0\} \times \mathbb{R}^n \setminus \{0\}$, by Lemma 5 (c), we have that $\| \cdot \|_{\mathcal{K}}$ and $\| \cdot \|_{\mathcal{K}^\circ}$ are differentiable. $F_{\mathcal{K}}$ and $F_{\mathcal{K}^\circ}$ are then also differentiable at $\{0\}$ because $\|\nabla F_{\mathcal{K}}(x)\|$ and $\|\nabla F_{\mathcal{K}^\circ}(d)\|$ converges to zero as $x$ and $d$ converge to zero (since $\nabla \| \cdot \|_{\mathcal{K}}(x)$ is of norm one). ∎

**Lemma 16 (Lower Bound on $\Theta$)** *Assume $\ell_1(r) \subset \mathcal{K} \subset \ell_\infty(R)$ for some $r, R > 0$. Let $x \in \mathcal{K}$ with $\|x\|_{\mathcal{K}} < 1$, $\eta > 0$ and $c \in \mathcal{K}^\circ$. Consider the realizations of random variable*

*$\xi \sim Ber(\|x\|_{\mathcal{K}})$, $i \sim \frac{1}{n}\mathbb{1}_n$, and $\epsilon \sim Rad(\frac{1}{2})$. We define $a \in \mathcal{K}$ (resp. $\tilde{c}$) similarly to $a_t$ (resp. $\tilde{c}_t$) in Algorithm 1 with*

$$a = \begin{cases} x/\|x\|_{\mathcal{K}} \text{ if } \xi = 1 \\ r\epsilon\ e_i \text{ otherwise,} \end{cases} \quad \text{and} \quad \tilde{c} = \frac{n}{r^2}(1-\xi)\frac{\langle a; c\rangle}{1-\|x\|_{\mathcal{K}}}a. \tag{5}$$

*Write $u = \nabla F_{\mathcal{K}}(x) - \eta\tilde{c}$ and $v = \nabla F_{\mathcal{K}}(x)$. Then, we have*

$$\frac{\|u\|_{\mathcal{K}^\circ} - \|v\|_{\mathcal{K}^\circ}}{1 + \|v\|_{\mathcal{K}^\circ}} \geq -\eta n\frac{R}{r}. \tag{6}$$

**Proof** Note that because $\ell_1(r) \subset \mathcal{K}$, we have $\pm re_i \in \mathcal{K}$ and in particular $a \in \mathcal{K}$. We now follow the argument of Bubeck and Cesa-Bianchi (2012). With the expression of $\nabla F_{\mathcal{K}}(x)$ in Lemma 15 and that $\big\|\nabla\|\cdot\|_{\mathcal{K}}(x)\big\|_{\mathcal{K}^\circ} = 1$ in Lemma 5, we have $\frac{1}{1+\|\nabla F_{\mathcal{K}}(x)\|_{\mathcal{K}^\circ}} = 1 - \|x\|_{\mathcal{K}}$. So with the triangle inequality, we have $\|v - \eta\tilde{c}\|_{\mathcal{K}^\circ} \geq \|v\|_{\mathcal{K}^\circ} - \eta\|\tilde{c}\|_{\mathcal{K}^\circ}$ so that we obtain

$$\frac{\|v - \eta\tilde{c}\|_{\mathcal{K}^\circ} - \|v\|_{\mathcal{K}^\circ}}{1 + \|v\|_{\mathcal{K}^\circ}} \geq -\eta\|\tilde{c}\|_{\mathcal{K}^\circ}(1 - \|x\|_{\mathcal{K}}).$$

Then, since $\tilde{c} = \dfrac{n}{r^2}(1-\xi)\dfrac{\langle a; c\rangle}{1-\|x\|_{\mathcal{K}}}a$, we have

$$\frac{\|v - \eta\tilde{c}\|_{\mathcal{K}^\circ} - \|v\|_{\mathcal{K}^\circ}}{1 + \|v\|_{\mathcal{K}^\circ}} \geq -\eta\frac{n}{r^2}(1-\xi)|\langle a; c\rangle| \cdot \|a\|_{\mathcal{K}^\circ}.$$

Because $\|\cdot\|_{\mathcal{K}}$ and $\|\cdot\|_{\mathcal{K}^\circ}$ are dual norms and $(a, c) \in \mathcal{K}\times\mathcal{K}^\circ$ we have $|\langle a; c\rangle| \leq \|a\|_{\mathcal{K}}\|c\|_{\mathcal{K}^\circ} \leq 1$, which leads to

$$\frac{\|v - \eta\tilde{c}\|_{\mathcal{K}^\circ} - \|v\|_{\mathcal{K}^\circ}}{1 + \|v\|_{\mathcal{K}^\circ}} \geq -\eta\frac{n}{r^2}\|a\|_{\mathcal{K}^\circ}.$$

When $\xi = 1$, (6) is already satisfied. Otherwise, $\xi = 0$ and by definition of $a$, we have $a = \epsilon re_i$ with $i \in [n]$ and $\epsilon \in \{-1, 1\}$. Since $\mathcal{K} \subset \ell_\infty(R)$, we have $\ell_\infty(R)^\circ = \ell_1(1/R) \subset \mathcal{K}^\circ$ and $e_i/R \in \mathcal{K}^\circ$. Hence, $\|re_i\|_{\mathcal{K}^\circ} = rR\|e_i/R\|_{\mathcal{K}^\circ} \leq rR$. So finally, we obtain

$$\frac{\|v - \eta\tilde{c}\|_{\mathcal{K}^\circ} - \|v\|_{\mathcal{K}^\circ}}{1 + \|v\|_{\mathcal{K}^\circ}} \geq -\eta n\frac{R}{r}.$$

■

The following lemma is instrumental to obtaining the pseudo-regret bounds. Note that the distance of $x_t$ to $\mathcal{K}$ is controlled by $\gamma$, see Line 11 in Algorithm 1. Finally, the sole difference with the bound obtained with the Euclidean ball is with the extra factor $1 + 4(\alpha+1)/\alpha$ and the constraint in $\eta$ that now depends on the ratio $r/R$ which, *e.g.*, equals 1 for any $\ell_q(1)$ ball.

**Lemma 17 (One Term Upper Bound Strong Convexity)** *Consider $\mathcal{K}$ a $\alpha$-strongly convex and centrally symmetric set with non-empty interior. Assume that $\ell_1(r) \subset \mathcal{K} \subset \ell_\infty(R)$ for some $r, R > 0$. Let $x \in \mathcal{K}$ s.t. $\|x\|_{\mathcal{K}} < 1$ and $\tilde{c}$ as defined in (5). If $0 < \eta \leq \frac{1}{2n}\frac{r}{R}$, then we have*

$$D_{F^*_{\mathcal{K}}}(\nabla F_{\mathcal{K}}(x) - \eta\tilde{c}, \nabla F_{\mathcal{K}}(x)) \leq (1 - \|x\|_{\mathcal{K}})\Big(1 + \frac{4(\alpha+1)}{\alpha}\Big)\eta^2\|\tilde{c}\|^2_{\mathcal{K}^\circ}.$$

**Proof** Let us write $u = \nabla F_{\mathcal{K}}(x) - \eta\tilde{c}$, $v = \nabla F_{\mathcal{K}}(x)$, and $\Theta = \frac{\|u\|_{\mathcal{K}^\circ} - \|v\|_{\mathcal{K}^\circ}}{1+\|v\|_{\mathcal{K}^\circ}}$. Elementary manipulations combined with Lemma 15 give

$$
\begin{aligned}
D_{F^*_{\mathcal{K}}}(u,v) &= F^*_{\mathcal{K}}(u) - F^*_{\mathcal{K}}(v) - \langle \nabla F^*_{\mathcal{K}}(v); u - v\rangle \\
&= \|u\|_{\mathcal{K}^\circ} - \|v\|_{\mathcal{K}^\circ} - \ln\Big(\frac{1+\|u\|_{\mathcal{K}^\circ}}{1+\|v\|_{\mathcal{K}^\circ}}\Big) - \frac{\|v\|_{\mathcal{K}^\circ}}{1+\|v\|_{\mathcal{K}^\circ}}\big\langle \nabla\|\cdot\|_{\mathcal{K}^\circ}(v); u-v\big\rangle \\
&= \|u\|_{\mathcal{K}^\circ} - \|v\|_{\mathcal{K}^\circ} - \ln\big(1+\Theta\big) - \frac{\|v\|_{\mathcal{K}^\circ}}{1+\|v\|_{\mathcal{K}^\circ}}\big\langle \nabla\|\cdot\|_{\mathcal{K}^\circ}(v); u-v\big\rangle \\
&= \frac{1}{1+\|v\|_{\mathcal{K}^\circ}}\Big[(1+\|v\|_{\mathcal{K}^\circ})(\|u\|_{\mathcal{K}^\circ} - \|v\|_{\mathcal{K}^\circ}) - (1+\|v\|_{\mathcal{K}^\circ})\ln\big(1+\Theta\big) \\
&\qquad\qquad - \|v\|_{\mathcal{K}^\circ}\big\langle \nabla\|\cdot\|_{\mathcal{K}^\circ}(v); u-v\big\rangle\Big] \\
&= \Theta - \ln\big(1+\Theta\big) + \frac{1}{1+\|v\|_{\mathcal{K}^\circ}}\underbrace{\Big[\|v\|_{\mathcal{K}^\circ}(\|u\|_{\mathcal{K}^\circ} - \|v\|_{\mathcal{K}^\circ}) - \|v\|_{\mathcal{K}^\circ}\big\langle \nabla\|\cdot\|_{\mathcal{K}^\circ}(v); u-v\big\rangle\Big]}_{H\triangleq}.
\end{aligned}
$$

Let us add and subtract $-\frac{1}{2}\|u\|^2_{\mathcal{K}^\circ}$ in $H$. We obtain

$$
H = \|v\|_{\mathcal{K}^\circ}\|u\|_{\mathcal{K}^\circ} - \frac{1}{2}\|v\|^2_{\mathcal{K}^\circ} - \frac{1}{2}\|u\|^2_{\mathcal{K}^\circ} + \frac{1}{2}\|u\|^2_{\mathcal{K}^\circ} - \frac{1}{2}\|v\|^2_{\mathcal{K}^\circ} - \big\langle \|v\|_{\mathcal{K}^\circ}\nabla\|\cdot\|_{\mathcal{K}^\circ}(v); u-v\big\rangle.
$$

We note that $\nabla\frac{1}{2}\|\cdot\|^2_{\mathcal{K}^\circ}(v) = \|v\|_{\mathcal{K}^\circ}\nabla\|\cdot\|_{\mathcal{K}^\circ}(v)$. It is then crucial to observe that the Bregman divergence of $\frac{1}{2}\|\cdot\|_{\mathcal{K}^\circ}$ appears as follows

$$
\begin{aligned}
H &= \|v\|_{\mathcal{K}^\circ}\|u\|_{\mathcal{K}^\circ} - \frac{1}{2}\|v\|^2_{\mathcal{K}^\circ} - \frac{1}{2}\|u\|^2_{\mathcal{K}^\circ} + D_{\frac{1}{2}\|\cdot\|^2_{\mathcal{K}^\circ}}(u,v) \\
&= -\frac{1}{2}\big(\|u\|_{\mathcal{K}^\circ} - \|v\|_{\mathcal{K}^\circ}\big)^2 + D_{\frac{1}{2}\|\cdot\|^2_{\mathcal{K}^\circ}}(u,v).
\end{aligned}
$$

Overall, with careful rewriting, we obtain that for any $(u,v)\in\mathbb{R}^n$

$$
D_{F^*_{\mathcal{K}}}(u,v) = \Theta - \ln\big(1+\Theta\big) - \frac{1}{2}\frac{\big(\|u\|_{\mathcal{K}^\circ} - \|v\|_{\mathcal{K}^\circ}\big)^2}{1+\|v\|_{\mathcal{K}^\circ}} + \frac{1}{1+\|v\|_{\mathcal{K}^\circ}}D_{\frac{1}{2}\|\cdot\|^2_{\mathcal{K}^\circ}}(u,v).
$$

With $\frac{1}{1+\|v\|_{\mathcal{K}^\circ}} = \frac{1}{1+\|\nabla F_{\mathcal{K}}(x)\|_{\mathcal{K}^\circ}} = 1 - \|x\|_{\mathcal{K}}$ (Lemma 15 and $\nabla\|\cdot\|_{\mathcal{K}}(x)$ is norm 1) it follows

$$
D_{F^*_{\mathcal{K}}}(u,v) \leq \Theta - \ln\big(1+\Theta\big) + (1-\|x\|_{\mathcal{K}})D_{\frac{1}{2}\|\cdot\|^2_{\mathcal{K}^\circ}}(u,v).
$$

Then, to upper bound $\Theta - \ln\big(1+\Theta\big)$, we note that $\ln(1+\theta) \geq \theta - \theta^2$ for all $\theta \geq -\frac{1}{2}$. Hence, we need to choose $\eta$ such that $\Theta \geq -\frac{1}{2}$. If $-\eta n\frac{R}{r} \geq -\frac{1}{2}$, *i.e.*, for $\eta \leq \frac{1}{2n}\frac{r}{R}$, Lemma 16 implies that $\Theta \geq -\frac{1}{2}$. Thus,

$$
D_{F^*_{\mathcal{K}}}(u,v) \leq \Big(\frac{\|u\|_{\mathcal{K}^\circ} - \|v\|_{\mathcal{K}^\circ}}{1+\|v\|_{\mathcal{K}^\circ}}\Big)^2 + (1-\|x\|_{\mathcal{K}})D_{\frac{1}{2}\|\cdot\|^2_{\mathcal{K}^\circ}}(u,v).
$$

Then, by the triangle inequality, and $1/(1+\|v\|_{\mathcal{K}^\circ}) = 1 - \|x\|_{\mathcal{K}}$, we have

$$
D_{F^*_{\mathcal{K}}}(u,v) \leq (1-\|x\|_{\mathcal{K}})^2\|u-v\|^2_{\mathcal{K}^\circ} + (1-\|x\|_{\mathcal{K}})D_{\frac{1}{2}\|\cdot\|^2_{\mathcal{K}^\circ}}(u,v). \tag{7}
$$

Then, with Corollary 8, we have $D_{\frac{1}{2}\|\cdot\|^2_{\mathcal{K}^\circ}}(u,v) \leq \frac{4(\alpha+1)}{\alpha}\|u-v\|^2_{\mathcal{K}^\circ}$. Hence by combining it with (7), we obtain

$$D_{F^*_{\mathcal{K}}}(u,v) \leq (1-\|x\|_{\mathcal{K}})\|u-v\|^2_{\mathcal{K}^\circ}\big[1+\frac{4(\alpha+1)}{\alpha}\big].$$

■

With the very same technique, we obtain another form of upper bound when the set is uniformly convex. For the sake of clarity we write it as a corollary of Lemma 17 although it is an extension.

**Corollary 18 (One Term Upper Bound Uniform Convexity)** *Let $q \geq 2$ and $p \in ]1,2]$ s.t. $1/p+1/q=1$. Consider $\mathcal{K}$ an $(\alpha,q)$-uniformly convex and centrally symmetric with non-empty interior set. Assume that $\ell_1(r) \subset \mathcal{K} \subset \ell_\infty(R)$ for some $r,R>0$. Let $x \in \mathcal{K}$ s.t. $\|x\|_{\mathcal{K}} < 1$ and $\tilde{c}$ as defined in (5). If $0 < \eta \leq \frac{1}{2n}\frac{r}{R}$, then we have*

$$D_{F^*_{\mathcal{K}}}(\nabla F_{\mathcal{K}}(x) - \eta\tilde{c}, \nabla F_{\mathcal{K}}(x)) \leq (1-\|x\|_{\mathcal{K}})\eta^p\|\tilde{c}\|^p_{\mathcal{K}^\circ}((1/2)^{2-p}+L),$$

*with $L \triangleq 2p(1+(q/(2\alpha))^{1/(q-1)})$.*

**Proof** The proof is exactly the same as Lemma 17 until (7). Here, by (1) in Lemma 7, we have $D_{\frac{1}{2}\|\cdot\|^2_{\mathcal{K}^\circ}}(u,v) \leq 2p\big(1+(q/(2\alpha))^{1/(q-1)}\big)\|u-v\|^p_{\mathcal{K}^\circ}$. Hence, we now have

$$\begin{aligned} D_{F^*_{\mathcal{K}}}(u,v) &\leq (1-\|x\|_{\mathcal{K}})^2\|u-v\|^2_{\mathcal{K}^\circ} + (1-\|x\|_{\mathcal{K}})2p\big(1+(q/(2\alpha))^{1/(p-1)}\big)\|u-v\|^p_{\mathcal{K}^\circ} \\ &\leq (1-\|x\|_{\mathcal{K}})\|u-v\|^p_{\mathcal{K}^\circ}\big[(1-\|x\|_{\mathcal{K}})\|u-v\|^{2-p}_{\mathcal{K}^\circ} + 2p\big(1+(q/(2\alpha))^{1/(q-1)}\big)\big]. \end{aligned}$$

We now simply need to bound the term $(1-\|x\|_{\mathcal{K}})\|u-v\|^{2-p}_{\mathcal{K}^\circ}$. We have $u-v=\eta\tilde{c}$, and by definition of $\tilde{c}$ in (5), when $\xi=0$, we have

$$(1-\|x\|_{\mathcal{K}})\|u-v\|^{2-p}_{\mathcal{K}^\circ} = (1-\|x\|_{\mathcal{K}})^{p-1}\Big[\frac{n\eta}{r^2}|\langle c; re_i\rangle|\cdot\|re_i\|_{\mathcal{K}^\circ}\Big]^{2-p}.$$

Then, since $\ell_1(r) \subset \mathcal{K}$, $re_i \in \mathcal{K}$ and $c \in \mathcal{K}^\circ$, we have $|\langle c; re_i\rangle| \leq 1$. Also, since $\mathcal{K} \subset \ell_\infty(R)$, we have $\ell_\infty(R)^\circ = \ell_1(1/R) \subset \mathcal{K}^\circ$ and $e_i/R \in \mathcal{K}^\circ$, hence $\|re_i\|_{\mathcal{K}^\circ} \leq rR$. Besides, by the choice of $\eta$, we have $n\eta \leq r/(2R)$. We now have (case $\xi=1$ is immediate) with $\eta \leq r/(2nR)$ and because $(1-\|x\|_{\mathcal{K}}) \leq 1$ and $p-1>0$

$$(1-\|x\|_{\mathcal{K}})\|u-v\|^{2-p}_{\mathcal{K}^\circ} \leq 1\cdot\Big[n\eta\frac{rR}{r^2}\Big]^{2-p} \leq \Big[\frac{r}{2R}\frac{rR}{r^2}\Big]^{2-p} = 1/2^{2-p}.$$

Finally, we obtain

$$D_{F^*_{\mathcal{K}}}(u,v) \leq (1-\|x\|_{\mathcal{K}})\|u-v\|^p_{\mathcal{K}^\circ}\big[(1/2)^{2-p}+2p\big(1+(q/(2\alpha))^{1/(q-1)}\big)\big].$$

■

## 4. Conclusion

When the action set is strongly convex, we design a barrier function leading to a bandit algorithm with pseudo-regret in $\tilde{\mathcal{O}}(\sqrt{nT})$. We hence drastically extend the family of action sets for which such pseudo-regret hold, which provides new elements for the open question from (Bubeck and Cesa-Bianchi, 2012, §5.5.). To our knowledge, a $\tilde{\mathcal{O}}(\sqrt{nT})$ bound was known only when the action set is a simplex or an $\ell_p$ ball with $p \in ]1, 2]$.

When the set is $(\alpha, q)$-uniformly convex with $q \geq 2$, in Theorems 11 and 12 we assume that $\ell_q(r)$ is contained in the action set $\mathcal{K}$. It is restrictive but allows us to first prove improved pseudo-regret bounds outside the explicit $\ell_p$ case. Removing this assumption is an interesting research direction. However, it is not clear that the current classical algorithmic scheme with a barrier function is best adapted to *practically* leverage the strong convexity of the action set. Indeed, in the case of online linear learning, Huang et al. (2017) show that the simple FTL allows obtaining accelerated regret bounds. Such projection-free schemes have several benefits, *e.g.*, computational efficiency (Combettes and Pokutta, 2021) but in the case of FTL they also do not require smoothness of the action set (Molinaro, 2020) as opposed to Algorithm 1 which requires it to ensure differentiability of $F_{\mathcal{K}}$ and $F_{\mathcal{K}^\circ}$ simultaneously. Besides, they also exhibit adaptive properties to unknown structural assumptions, *e.g.*, unknown parameters of Hölderian Error Bounds (Kerdreux et al., 2019; Kerdreux, 2020).

At a high level, this work is an example of the favorable dimension-dependency of the sets' uniform convexity assumptions for the pseudo-regret bounds. It is crucial for large-scale machine learning. Such observations have already been made, *e.g.*, in constrained optimization (Polyak, 1966; Demyanov and Rubinov, 1970; Dunn, 1979; Kerdreux et al., 2021a,c), when the sets' $\alpha$-strong convexity leads to linear convergence rates of the Frank-Wolfe methods with a conditioning on the set that does not depend on the dimension. On the contrary, the linear convergence regimes for *corrective* versions of Frank-Wolfe on polytope with strongly convex functions suffer large dimension dependency, see, *e.g.*, (Lacoste-Julien and Jaggi, 2015; Diakonikolas et al., 2020; Garber, 2020; Carderera et al., 2021). This difference between polytope structures and uniform convexity assumption is even more apparent with infinite-dimensional constraints. Besides, to our knowledge, the uniform convexity structures for the sets are much less developed and understood than their functional counterpart, see, *e.g.*, (Kerdreux et al., 2021b). Arguably, this stems from a tendency in machine learning to consider that constraints are theoretically interchangeable with penalization. It is often not quite accurate in terms of convergence results and the algorithmic strategies developed differ. The linear bandit setting is a simple example where such symmetry is structurally not relevant.

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## Appendix A. Consequences of Set Strong Convexity

We provide here a simplification of (Kerdreux et al., 2021b, Theorem 4.1.), see also (Borwein et al., 2009). Let us first recall the *scaling inequality* that provides an equivalent characterization of uniformly convex sets (Kerdreux et al., 2021b, Theorem 4.1.). These inequalities quantify the behavior of the normal cone directions at the boundary of $\mathcal{K}$. As such, they offer a geometrical intuition on uniform convex as opposed to the algebraic Definition 4. Also, they are useful to prove Theorem 20.

**Lemma 19 (Scaling Inequality)** *Let $\alpha > 0$ and $q \geq 2$. Assume $\mathcal{K}$ is $(\alpha, q)$-uniformly convex. Then, for any $x, y \in \mathcal{K} \times \partial\mathcal{K}$ and $d \in N_{\mathcal{K}}(y)$, we have*

$$\langle d; y - x\rangle \geq \frac{\alpha}{q}\|x - y\|_{\mathcal{K}}^q \|d\|_{\mathcal{K}^\circ}. \tag{8}$$

**Proof** We repeat the proof for completeness. Let $(x, y, d)$ as in the lemma. In particular, $y \in \mathrm{argmax}_{v\in\mathcal{K}}\langle d; v\rangle$. By optimality of $y$ and uniform convexity of $\mathcal{K}$, for any $\gamma \in ]0, 1[$ and $z$ with $\|z\|_{\mathcal{K}} \leq 1$ we have

$$\langle d; y\rangle \geq \langle d; \gamma x + (1 - \gamma)y + \frac{\alpha}{q}\gamma(1 - \gamma)\|x - y\|_{\mathcal{K}}^q z\rangle.$$

After simplification, we obtain for any $\gamma \in ]0, 1[, z \in \mathcal{K}$

$$\langle d; y - x\rangle \geq \frac{\alpha}{q}(1 - \gamma)\|y - x\|_{\mathcal{K}}^q\langle d; z\rangle.$$

Hence, by definition of the dual norm of $\|\cdot\|_{\mathcal{K}}$ and $\|\cdot\|_{\mathcal{K}}^\star = \|\cdot\|_{\mathcal{K}^\circ}$, we obtain

$$\langle d; y - x\rangle \geq \frac{\alpha}{q}\|y - x\|_{\mathcal{K}}^q\|d\|_{\mathcal{K}^\circ}.$$

■

Theorem 20 is slightly different from (Kerdreux et al., 2021b, Theorem 4.1.) because we are interested in the smoothness property of $\frac{1}{2}\|\cdot\|_{\mathcal{K}^\circ}^2$ instead of $\frac{1}{q}\|\cdot\|_{\mathcal{K}^\circ}^q$ when the set $\mathcal{K}$ is $(\alpha, q)$-uniformly convex. The proof is however very similar. The main different is that in (Kerdreux et al., 2021b, Theorem 4.1.) the smoothness property was ensured on $\mathbb{R}^n$ while here it is only true on bounded domains like $\mathcal{K}^\circ$.

**Theorem 20** *Let $\alpha > 0$, $q \geq 2$ and $p \in ]1, 2]$ s.t $1/p + 1/q = 1$. Consider $\mathcal{K} \subset \mathbb{R}^n$ a centrally symmetric compact convex with non-empty interior. Assume $\mathcal{K}$ is smooth and $(\alpha, q)$-uniformly convex w.r.t. $\|\cdot\|_{\mathcal{K}}$ (Definition 4), then*

$$\frac{1}{2}\|\cdot\|_{\mathcal{K}^\circ}^2 \text{ is } (L, p)\text{-Hölder Smooth on } \mathcal{K}^\circ,$$

*with*

$$L = 2p\Big(1 + \Big(\frac{q}{2\alpha}\Big)^{1/(q-1)}\Big).$$

**Proof** The proof follows (Kerdreux et al., 2021b, Theorem 4.1). We repeat it to obtain quantitative results. The proof proceed is two steps: first prove the Hölder-smoothness of $\|\cdot\|_{\mathcal{K}^\circ}$ on $\partial\mathcal{K}^\circ$ and then prove the Hölder-smoothness of $\frac{1}{2}\|\cdot\|_{\mathcal{K}^\circ}^2$ on $\mathcal{K}^\circ$.

**Smoothness of $\|\cdot\|_{\mathcal{K}^\circ}$ on $\partial\mathcal{K}^\circ$.** Let $(d_1, d_2) \in \partial\mathcal{K}^\circ \times \partial\mathcal{K}^\circ$ and $(x_1, x_2) \in \partial\mathcal{K} \times \partial\mathcal{K}$ s.t. $x_i \in \text{argmax}_{x\in\mathcal{K}}\langle d_i; x\rangle$ for $i = 1, 2$. Because $\mathcal{K}$ is strictly convex (uniform convexity implies strict convexity), the $x_i$ are unique and by Lemma 5, $\nabla\|\cdot\|_{\mathcal{K}^\circ}(d_i) = x_i$ for $i = 1, 2$. Note that equivalently we have $d_i \in N_{\mathcal{K}}(x_i)$. Applying the scaling inequalities (8) we have for any $x \in \mathcal{K}$

$$\begin{cases} \langle d_1; x_1 - x\rangle \geq \alpha/q\|d_1\|_{\mathcal{K}^\circ} \cdot \|x_1 - x\|_{\mathcal{K}}^q = \alpha/q\|x_1 - x\|_{\mathcal{K}}^q \\ \langle d_2; x_2 - x\rangle \geq \alpha/q\|d_2\|_{\mathcal{K}^\circ} \cdot \|x_2 - x\|_{\mathcal{K}}^q = \alpha/q\|x_2 - x\|_{\mathcal{K}}^q. \end{cases}$$

Then, by summing the two inequalities evaluated respectively at $x = x_2$ and $x = x_1$, we have

$$\langle d_1 - d_2; x_1 - x_2\rangle \geq 2\alpha/q\|x_1 - x_2\|_{\mathcal{K}}^q.$$

By Cauchy-Schwartz, we obtain

$$\big\|d_1 - d_2\big\|_{\mathcal{K}^\circ} \cdot \big\|\nabla\|\cdot\|_{\mathcal{K}^\circ}(d_1) - \nabla\|\cdot\|_{\mathcal{K}^\circ}(d_2)\big\|_{\mathcal{K}} \geq 2\alpha/q\big\|\nabla\|\cdot\|_{\mathcal{K}^\circ}(d_1) - \nabla\|\cdot\|_{\mathcal{K}^\circ}(d_2)\big\|_{\mathcal{K}}^q,$$

and conclude that

$$\big\|\nabla\|\cdot\|_{\mathcal{K}^\circ}(d_1) - \nabla\|\cdot\|_{\mathcal{K}^\circ}(d_2)\big\|_{\mathcal{K}} \leq \frac{1}{(2\alpha/q)^{1/(q-1)}}\big\|d_1 - d_2\big\|_{\mathcal{K}^\circ}^{1/(q-1)}. \tag{9}$$

**Smoothness of $\frac{1}{2}\|\cdot\|_{\mathcal{K}^\circ}^2$ on $\mathcal{K}^\circ$.** Let us first note that $\nabla\frac{1}{2}\|\cdot\|_{\mathcal{K}^\circ}^2(d) = \|d\|_{\mathcal{K}^\circ}\nabla\|\cdot\|_{\mathcal{K}^\circ}(d)$. Hence, since $\|\cdot\|_{\mathcal{K}^\circ}(d)$ is norm 1, when $d$ approaches $0_n$, the limit of $\nabla\frac{1}{2}\|\cdot\|_{\mathcal{K}^\circ}^2(d)$ is 0 and hence $\frac{1}{2}\|\cdot\|_{\mathcal{K}^\circ}^2$ is differentiable on $\mathbb{R}^n$ (as opposed to $\|\cdot\|_{\mathcal{K}^\circ}$ that is not differentiable at 0).

Similarly, consider non-zeros $(d_1, d_2) \in \mathcal{K}^\circ \times \mathcal{K}^\circ$ and the $(x_1, x_2) \in \partial\mathcal{K} \times \partial\mathcal{K}$ s.t. $x_i \in \text{argmax}_{x\in\mathcal{K}}\langle d_i; x\rangle$ for $i = 1, 2$. Because of (b) in Lemma 5, we have $\nabla\|\cdot\|_{\mathcal{K}^\circ}(d_1) = \nabla\|\cdot\|_{\mathcal{K}^\circ}(d_1/\|d_1\|_{\mathcal{K}^\circ})$. Hence, with (9), we obtain

$$\big\|\nabla\|\cdot\|_{\mathcal{K}^\circ}(d_1) - \|\cdot\|_{\mathcal{K}^\circ}(d_2)\big\|_{\mathcal{K}} \leq \frac{1}{(2\alpha/q)^{1/(q-1)}}\big\|d_1/\|d_1\|_{\mathcal{K}^\circ} - d_2/\|d_2\|_{\mathcal{K}^\circ}\big\|_{\mathcal{K}^\circ}^{1/(q-1)}.$$

Write $C \triangleq 1/(2\alpha/q)^{1/(q-1)}$ and $I \triangleq \big\|\nabla\frac{1}{2}\|\cdot\|_{\mathcal{K}^\circ}^2(d_1) - \frac{1}{2}\nabla\|\cdot\|_{\mathcal{K}^\circ}^2(d_2)\big\|_{\mathcal{K}}$. Let us now consider

$$\begin{aligned} I &= \big\|\|d_1\|_{\mathcal{K}^\circ}\nabla\|\cdot\|_{\mathcal{K}^\circ}(d_1) - \|d_2\|_{\mathcal{K}^\circ}\nabla\|\cdot\|_{\mathcal{K}^\circ}(d_2)\big\|_{\mathcal{K}} \\ &\leq \|d_1\|_{\mathcal{K}^\circ}\big\|\nabla\|\cdot\|_{\mathcal{K}^\circ}(d_1) - \nabla\|\cdot\|_{\mathcal{K}^\circ}(d_2)\big\| + \big\|\nabla\|\cdot\|_{\mathcal{K}^\circ}(d_2)\big\| \cdot \big|\|d_1\|_{\mathcal{K}^\circ} - \|d_2\|_{\mathcal{K}^\circ}\big| \\ &\leq C\|d_1\|_{\mathcal{K}^\circ}^{1+1/(q-1)}\|d_1\|_{\mathcal{K}^\circ}^{1/(q-1)}\big\|d_1/\|d_1\|_{\mathcal{K}^\circ} - d_2/\|d_2\|_{\mathcal{K}^\circ}\big\|_{\mathcal{K}^\circ}^{1/(q-1)} + \|d_1 - d_2\|_{\mathcal{K}^\circ} \\ &\leq C\|d_1\|_{\mathcal{K}^\circ}^{q/(q-1)}\big\|d_1 - d_2\|d_1\|_{\mathcal{K}^\circ}/\|d_2\|_{\mathcal{K}^\circ}\big\|_{\mathcal{K}^\circ}^{1/(q-1)} + \|d_1 - d_2\|_{\mathcal{K}^\circ}^{q/(q-1)}\|d_1 - d_2\|_{\mathcal{K}^\circ}^{1/(q-1)}. \end{aligned}$$

For $i = 1, 2$, $d_i \in \mathcal{K}^\circ$ so that $\|d_i\|_{\mathcal{K}^\circ} \leq 1$. We then obtain

$$I \leq C\big\|d_1 - d_2\|d_1\|_{\mathcal{K}^\circ}/\|d_2\|_{\mathcal{K}^\circ}\big\|_{\mathcal{K}^\circ}^{1/(q-1)} + 2\|d_1 - d_2\|_{\mathcal{K}^\circ}^{1/(q-1)}.$$

Also, with the triangle inequality

$$\big\|d_1 - d_2\frac{\|d_1\|_{\mathcal{K}^\circ}}{\|d_2\|_{\mathcal{K}^\circ}}\big\| \leq \|d_1 - d_2\|_{\mathcal{K}^\circ} + \big\|d_2 - d_2\frac{\|d_1\|_{\mathcal{K}^\circ}}{\|d_2\|_{\mathcal{K}^\circ}}\big\| \leq \|d_1 - d_2\|_{\mathcal{K}^\circ} + \|d_2\|_{\mathcal{K}^\circ} - \|d_1\|_{\mathcal{K}^\circ} \leq 2\|d_1 - d_2\|_{\mathcal{K}^\circ}.$$

Hence, we finally obtain

$$\big\|\nabla\frac{1}{2}\|\cdot\|^2_{\mathcal{K}^\circ}(d_1) - \frac{1}{2}\nabla\|\cdot\|^2_{\mathcal{K}^\circ}(d_2)\big\|_{\mathcal{K}} \leq 2(C+1)\big\|d_1 - d_2\big\|_{\mathcal{K}^\circ}^{1/(q-1)}.$$

This equivalently means that $\frac{1}{2}\|\cdot\|^\circ_{\mathcal{K}}$ is $\big(2(C+1), 1+1/(q-1)\big)$-Hölder smooth as defined in (Hölder-Smoothness). Hence, since $q-1 = 1/(p-1)$, we get that $\frac{1}{2}\|\cdot\|^\circ_{\mathcal{K}}$ is $\big(2p(C+1), p\big)$-Hölder smooth. ■