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Thompson Sampling Guided Stochastic Searching on the Line for Deceptive Environments with Applications to Root-Finding Problems*

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Abstract

The multi-armed bandit problem forms the foundation for solving a wide range of online stochastic optimization problems through a simple, yet effective mechanism. One simply casts the problem as a gambler who repeatedly pulls one out of N slot machine arms, eliciting random rewards. Learning of reward probabilities is then combined with reward maximization, by carefully balancing reward exploration against reward exploitation. In this paper, we address a particularly intriguing variant of the multi-armed bandit problem, referred to as the *Stochastic Point Location (SPL)* problem. The gambler is here only told whether the optimal arm (point) lies to the "left" or to the "right" of the arm pulled, with the feedback being erroneous with probability $1 - \pi$. This formulation thus targets optimization in continuous action spaces with both *informative* and *deceptive* feedback. To tackle this class of problems, we formulate a compact and scalable Bayesian representation of the solution space that simultaneously captures both the location of the optimal arm as well as the probability of receiving correct feedback. We further introduce the accompanying Thompson Sampling guided Stochastic Point Location (TS-SPL) scheme for balancing exploration against exploitation. By learning π, TS-SPL also supports *deceptive* environments that are lying about the direction of the optimal arm. This, in turn, allows us to address the fundamental Stochastic Root Finding (SRF) problem. Empirical results demonstrate that our scheme deals with both deceptive and informative environments, significantly outperforming competing algorithms both for SRF and SPL.

Keywords: thompson sampling, searching on the line, probabilistic bisection search, deceptive environment, stochastic point location

1. Introduction

Research on the *Stochastic Point Location* (SPL) problem (Oommen, 1997) has delivered increasingly efficient schemes for locating the optimal point on a line. In all brevity, the optimal point must be found based on iteratively proposing candidate points, with each candidate revealing whether the optimal point lies to the candidate's left or to its right. The provided directions can be erroneous, and the goal is to locate the optimal point with

[*] A preliminary version of some of the results of this paper appears in the Proceedings of AIAI'15.

as few non-optimal candidate proposals as possible. The SPL problem can also be cast as an agent that moves on a line, attempting to locate a particular location λ^*. The agent communicates with a teacher that notifies the agent whether its current location λ is greater or lower than λ^*. However, the teacher is of a stochastic nature and feeds the agent erroneous feedback with probability $1 - \pi$.

Despite the simplicity of the SPL problem, SPL schemes have provided novel solutions for a wide range of problems. Intriguing applications include the estimation of non-stationary binomial distributions (Yazidi et al., 2012b), communication network routing (Oommen et al., 2007), and meta-optimization (Oommen et al., 2009). Furthermore, recent research that addresses the related *Stochastic Root-Finding* (SRF) problem provides promising solutions for parameter estimation, transportation system optimization, as well as supply chain optimization (Chen and Schmeiser, 2001; Pasupathy and Kim, 2011).

State-of-the-art. Adaptive Step Searching (ASS) (Tao et al., 2013) is currently the leading approach to solving SPL problems, although it is outperformed by Hierarchical Stochastic Searching on the Line (HSSL) (Yazidi et al., 2012a) in highly volatile non-stationary environments (Tao et al., 2013). Optimal Computing Budget Allocation (OCBA) has also been applied to SPL (Zhang et al., 2015) and provides stable solutions while converging slightly slower than ASS. Unfortunately, these state-of-the-art schemes fail when noise increases beyond a certain degree, which happens when the majority of obtained directions mislead rather than guide. Indeed, by naively following the directions provided under such circumstances, one is systematically led away from the optimal point. We refer to these kinds of problem environments as *deceptive* environments, as opposed to *informative* ones, which are explained in more detail below.

To the best of the authors' knowledge, the pioneering CPL-AdS (Oommen et al., 2003) scheme was the first known approach handling deceptive SPL environments. CPL-AdS relies on two consecutive phases. In the first phase, a sequence of intelligently selected questions is used to classify the environment as either informative or deceptive. By spending a sufficient amount of time in this phase, the classification can be made arbitrarily accurate. In the second phase, a regular SPL scheme is applied, except that the directions obtained are reversed if the problem environment was classified as deceptive in the first phase. This means that the scheme may have to remain in the first phase for an extensive amount of time to ensure that the problem environment is correctly classified, otherwise, one risks being systematically misled in the second phase. These properties largely render CPL-AdS inappropriate for online or anytime problem solving.

Recently, HSSL has been extended by Zhang et al. to cover both informative and deceptive environments, using a Symmetric HSSL (SHSSL) scheme (Zhang et al., 2016). This scheme essentially runs two HSSL schemes in conjunction: one regular, which handles informative environments, and one that inverts all feedback from the environment, to handle deceptive environments. The hierarchy navigation capabilities of HSSL are then exploited to allow SHSSL to switch between the two HSSLs, depending on the nature of the environment. However, a significant limitation of HSSL, namely, that π must be larger than the conjugate of the golden ratio, carries over to SHSSL. Indeed, SHSSL fails to converge for $\pi \in [0.382, 0.682]$, which amounts to approximately 30% of the feasible values for π. This is in contrast to the approach we propose in this paper, as well as to CPL-AdS (Oommen et al., 2003), since both of these schemes operate along the whole range of π (apart from $\pi = 0.5$).

To cast further light on the challenges lined out above, we here introduce the *N-Door Puzzle* as a framework for modeling deception. We also propose an accompanying novel solution scheme — *Thompson Sampling guided Stochastic Point Location* (TS-SPL). The TS-SPL scheme handles both SPL and SRF problems, and is capable of *simultaneously* solving the problem as well as determining whether we are dealing with an informative or a deceptive environment. As we shall see, not only does this scheme handle an arbitrary level of noise, but it also outperforms current state-of-the-art techniques in both informative and deceptive environments.

The N-Door Puzzle. In the book "To Mock a Mockingbird" (Smullyan, 1988) the following puzzle is formulated: *"Someone was sentenced to death, but since the king loves riddles, he threw this guy into a room with two doors. One leading to death, one leading to freedom. There are two guards, each one guarding one door. One of the guards is a perfect liar, the other one will always tell the truth. The man is allowed to ask one guard a single yes-no question and then has to decide, which door to take. What single question can he ask to guarantee his freedom?"* To avoid spoiling the puzzle for the reader, we omit the solution here and note that asking a double negative question will often be the correct course of action for these types of puzzles.

The above puzzle can be generalized by increasing the number of doors. Instead of deciding between merely two doors, the prisoner now faces N doors, with a guard posted between each pair of doors. Only a single door leads to freedom, the remaining doors lead to death. Every day at sunrise, the prisoner is allowed to ask one of the guards whether the door leading to freedom is to the left or to the right of the guard. However, only a fixed proportion of the guards answers truthfully, the rest are compulsive liars. Further, the guards are randomly assigned a position at each sunrise, and thus, knowing who lies and who tells the truth is impossible. As an additional complication, depending on the mood of the king, the prisoner may be ordered to walk through a door of his choice at an arbitrary day. Therefore, to save his life, it is imperative that the prisoner determines as quickly as possible which door leads to freedom.

Specifically, let $\pi = \frac{\#\text{truthful guards}}{\#\text{guards}}$ be the fraction of truthful guards. Since the guards are randomly assigned a position each day, the probability of obtaining a truthful answer is governed by π. If $\pi < 0.5$ then the majority of the guards are compulsive liars, and the guards as an entity can be characterized as being *deceptive.* Conversely, if $\pi > 0.5$ then the majority of the guards are truthful and the guards can be seen as *informative.* For completeness, we mention that the puzzle is unsolvable for the case where π is exactly equal to $\frac{1}{2}$, since it is then impossible to obtain any information on neither the nature of the doors nor the guards.

Thompson Sampling. The Thompson Sampling (TS) principle was introduced by Thompson already in 1933 (Thompson, 1933) and now forms the basis for several state-of-the-art approaches to the Multi-Armed Bandit (MAB) problem — a fundamental sequential resource allocation problem that has challenged researchers for decades. At each time step in the MAB problem, one is offered to pull one out of N bandit arms, which in turn triggers a stochastic reward. Each arm has an underlying probability of providing a reward, however, these probabilities are unknown to the decision maker. The challenge is thus to decide which of the arms to pull at each time step, to maximize the expected total number of rewards obtained (Bubeck and Cesa-Bianchi, 2012).

In all brevity, TS seeks to achieve the above goal by quickly shifting from exploring reward probabilities to maximizing the number of rewards obtained. This is achieved by recursively estimating the underlying reward probability of each arm, using Bayesian filtering of the rewards obtained so far. TS then simply selects the next arm to pull based on the Bayesian estimates of the reward probabilities (one reward probability density function per arm).

The arm selection strategy of TS is rather straightforward, yet surprisingly efficient. To determine which arm to pull, a single candidate reward probability is sampled from the probability density function of each arm. *The arm with the highest sample value is the one pulled next.* The outcome of pulling this arm is in turn used to perform the next Bayesian update of the arm's reward probability estimate. It is this simple scheme that makes TS select arms with frequency proportional to the posterior probability of being optimal, leading to quick convergence towards always selecting the optimal arm.

TS has turned out to be among the top performers for traditional MAB problems (Granmo, 2010; Chapelle and Li, 2011), supported by theoretical regret bounds (Agrawal and Goyal, 2012, 2013a; Dong and Van Roy, 2018). It has also been successfully applied to contextual MAB problems (Agrawal and Goyal, 2013b), constrained Gaussian process optimization (Glimsdal and Granmo, 2013), distributed quality of service control in wireless networks (Granmo and Glimsdal, 2013), cognitive radio optimization (Jiao et al., 2016), as well as a foundation for solving the maximum a posteriori estimation problem (Tolpin and Wood, 2015).

Pure Exploration Bandits. Throughout this paper we assume that each SPL problem potentially takes part in a larger system consisting of multiple SPL problems, and not necessarily operating in isolation. From existing applications in the literature, such as web crawler load balancing (Granmo et al., 2007), it is clear that the value of an SPL scheme does hinge upon its ability to cooperate and interact with other decision makers. Such cooperation demands predictable behaviour from the individual decision makers, as well as coordinated balancing of exploring new solution candidates against maintaining good solution candidates. Without such an ability, the system as a whole will not be able to systematically move towards the more promising areas of the search space, gradually focusing in on an optimal configuration. Therefore, in this paper we omit a direct comparison with schemes that rely on a "fixed sampling *then* decide" approach, such as unimodal bandits (Jia and Mannor, 2011). For the same reason, we will not investigate purely exploitative bandits (Even-Dar et al., 2006; Jamieson et al., 2014; Audibert and Bubeck, 2010; Gabillon et al., 2011; Karnin et al., 2013), bandits that have a predefined finite time horizon and whose performance is only measured at the end of that horizon. Such algorithms are free to explore without any negative impact, and this allows them to outperform traditional exploitation-exploration bandits such as TS and UCB in scenarios where exploitation is not required.[1]

[1] There also exists a wide spectrum of techniques and schemes in the literature on the topic of searching with noise. See for instance (Pelc, 2002) for a comprehensive survey. These are unable to handle unknown and deceptive environments, with stochastic directional feedback, and are therefore not directly

Paper Contributions. In this paper, we introduce a novel scheme for solving the SPL problem, namely, TS-SPL. At the core of TS-SPL, we find a compact and scalable Bayesian representation of the SPL solution space. This Bayesian representation simultaneously captures both the location of the optimal point (bandit arm) as well as the probability of receiving correct feedback. We further introduce an accompanying scheme for balancing exploration against exploitation, based on TS. By learning π, TS-SPL also supports *deceptive* environments that are lying about the direction of the optimal arm. This, in turn, allows us to solve the fundamental SRF problem. More specifically, the contributions of the paper can be summarized as follows:

1. We introduce a novel TS-SPL scheme that represents the solution space of N-Door Puzzles, and thus SPL problems, in terms of a Bayesian model. As opposed to competing solutions that merely maintain and refine a single candidate solution, our Bayesian model encompasses the complete space of candidate solutions at every time instant.

2. We formulate a compact and scalable Bayesian representation of the solution space that simultaneously captures both the location of the optimal point (arm), as well as the probability of receiving correct feedback. This Bayesian representation of the problem opens up for efficient exploration and exploitation of the solution space with TS.

3. We link TS-SPL to so-called stochastic bisection search; and unify accompanying methods under the umbrella of TS.

4. Similarly, we enhance the Soft Generalized Binary Search (SGBS), Probabilistic Bisection Search (PBS) and Burnashev-Zigangirov Algorithm (BZ) by introducing novel parameter free solutions that take advantage of our Bayesian model of the N-door puzzle and the SPL problem. This approach eliminates previous reliance on knowing the exact degree of noise affecting the system to be optimized.

5. We provide the first unified empirical comparison of the key state-of-the-art SPL- and SRF solvers.

6. We finally demonstrate the empirical performance of TS-SPL for both SPL and SRF problems. TS-SPL outperforms the state-of-the-art algorithms in both informative and deceptive environments, except for the SGBS and BZ schemes with correctly specified observation noise.

Paper Outline. The paper is organized as follows. In Section 2, we present our scheme for TS guided SPL (TS-SPL). We first introduce the Bayesian model of the N-door puzzle. Based on the Bayesian model, we then formulate our TS-based scheme that balances solution space exploration against reward maximization. We further extend selected state-of-the-art solution schemes with our Bayesian N-door puzzle model. This extension removes the need for knowing the observation noise beforehand. In Section 3, we provide extensive empirical results comparing TS-SPL with state-of-the-art schemes for both SPL and SRF. We conclude in Section 4 and point to promising directions for further work.

comparable to SPL solution schemes. We have therefore not included this class of techniques in the present paper.

2. Thompson Sampling Guided Stochastic Point Location

In this section, we introduce the TS-SPL scheme. The scheme can be summarized as follows. At the core of TS-SPL we find a Bayesian model of the N-Door Puzzle. Formally, we represent an instance of the N-door puzzle as a tuple $(\lambda^*, \pi^*) \in D \times T$, where $D = \{d_1, \ldots, d_N\}$ is the set of doors and $T \in [0, 1]$ is the truthfulness of the guards. Let (λ^*, π^*) be the particular N-door puzzle faced. A novel aspect of TS-SPL is that instead of maintaining a single or a limited set of candidate solutions, we instead maintain a posterior distribution over the whole solution space, $(\lambda, \pi) \in D \times T$. This distribution is conditioned on the feedback already obtained up to time step n, allowing us to single in on (λ^*, π^*) as the number of time steps increases, ultimately converging to (λ^*, π^*).

Assuming no prior information, we assign a uniform distribution over $D \times T$, i.e., all puzzle instances are equally probable. By gradually refining the posterior distribution over $D \times T$, we can select guards to question in a goal-oriented manner. In all brevity, we sample a solution candidate (λ^c, π^c) from $D \times T$, selecting the guard to the left or to right of λ^c. The answer of the selected guard is then used to update our posterior distribution. By repeating this procedure, the expected probability of the underlying N-door puzzle instance (λ^*, π^*) increases monotonically, reducing the probability of other puzzle instances. In effect, given enough iterations, TS-SPL will correctly identify the door leading to freedom as the posterior probability of (λ^*, π^*) approaches unity.

2.1. Bayesian Model of the N-Door Puzzle

The main purpose of the Bayesian model is to facilitate the efficient calculation of a posterior distribution over the possible N-door puzzle instances, $D \times T$. Since the prisoner does not initially know which problem instance he is facing, and since the observations are stochastic, we cast D and T as two random variables. We further assume that D and T are independent of each other. Furthermore, the information we obtain from questioning the guards is represented as a set of random variables $Q = \{Q_1, \ldots, Q_n\}$, with each random variable Q_k representing the answer from question k. Finally, we assume that the outcomes of the individual questions $Q_k \in Q$ are independent when conditioned on D and P. For each question Q_k, we can then compute the probability of the answer ("left" or "right") that we received from the guard, as summarized in Table 1.

Guard to the left of door to freedom	$P(\text{left} \mid \text{guard, door, } t) = t$
	$P(\text{right} \mid \text{guard, door, } t) = 1 - t$
Guard to the right of door to freedom:	$P(\text{left} \mid \text{guard, door, } t) = 1 - t$
	$P(\text{right} \mid \text{guard, door, } t) = t$

Table 1: Conditional door probabilities

As an example, let us assume that the truthfulness of the guards is $t = 0.75$. If for instance the guard to the left of door d_4 replies that the door leading to freedom lies to his left, we can infer that all doors to the left have the likelihood of $t = 0.75$ of leading to freedom, and all the doors to the right have the likelihood $1 - t = 0.25$ of leading to freedom.

Applying Bayes Theorem to $P(Q|d,t)$, defined in Table 1, we are able to derive closed-form expressions for the posterior distributions of both D and T. The derivation of $P(d|Q)$, $d \in D$, follows [the derivation of $P(t|Q)$, $t \in T$, is analogous, and is left out here for the sake of brevity]:

$$\begin{aligned} P(d|Q) &= \sum_{t \in T} P(d,t|Q) && (1) \\ &\propto \sum_{t \in T} P(Q|d,t)P(d,t) && (2) \\ &= \sum_{t \in T} P(Q|d,t)P(d)P(t) && (3) \\ &= \sum_{t \in T} \hat{Q}Q^{+}P(d)P(t) && (4) \end{aligned}$$

Above, $\hat{Q} = \prod_{k=1}^{n-1} P(Q_k|d,t)$ and $Q^{+} = P(Q_n|d,t)$. Further, (2) follows directly from Bayes Theorem. We obtain (3) as a result of the independence of D and T, and (4) from the independence between the questions in Q. This leads to the following two equations for updating our knowledge about both the door probabilities (5) and the truthfulness of the guards (6).

$$P(d|Q) \propto \sum_{t \in T} \hat{Q}Q^{+}P(d)P(t) \qquad (5)$$

$$P(t|Q) \propto \sum_{d \in D} \hat{Q}Q^{+}P(d)P(t) \qquad (6)$$

2.2. Guard Selection

We have now formally determined how we can transform information from the guards into a probability distribution over which door leads to freedom. However, as mentioned previously, we also face a trade-off between exploring different doors and zeroing in on the best door found so far. To handle this trade-off, we model the door selection as a so-called Global Information MAB (GI-MAB) (Atan et al., 2015).

To decide which door should be selected at each iteration, we solve the GI-MAB by utilizing the principle of TS. Here, the selection process is simply to select a random door proportional to the probability that this door is the one that leads to freedom. Once the door has been selected, we need to decide which of the guards to query: the guard to the left or to the right of the door selected. We do this by randomly selecting one of the guards, again proportional to the sum of the probabilities of the doors next to each guard. Let us assume for instance that we have three doors $d_k, 1 \leq k \leq 3$ with the probabilities of leading to freedom: $P(d_1) = 0.1, P(d_2) = 0.2, P(d_3) = 0.7$. Then, according to the TS principle, these are also the probabilities we use to sample a particular door. Note that since the answer obtained from each guard affects the complete probability distribution over D (the probability associated with every door is updated), we have a GI-MAB as opposed to a traditional MAB.

2.3. Improving State-of-the-Art Schemes with the Bayesian Model of the N-Door Puzzle

A main advantage of TS-SPL compared to similar schemes is the utilization of the Bayesian model that enables TS-SPL to operate without knowing the problem parameters in advance. Due to TS-SPL's close connection to the Probabilistic Bisection Search (PBS) (Horstein, 1963), Noisy Generalized Binary Search (NGBS) (Nowak, 2008) and the BZ algorithm (Burnashev and Zigangirov, 1974), we will here use our Bayesian TS-SPL model to also make these other schemes parameter free.

Probabilistic Bisection Search

The goal of PBS[2] (Waeber et al., 2013; Nowak, 2008) is to locate an unknown point $X^* \in [0, 1]$. To acquire intelligence on the location of X^*, one queries an oracle of the relation between a point x and X^*. The oracle responds by informing whether x is on the left or the right side of X^*. If we assume that the oracle is always telling the truth, then the well-known deterministic bisection search, which halves the search space with each query, can be employed to efficiently find X^*. However, in PBS we assume that the Oracle provides correct answers with probability $p \in (0.5, 1.0]$ and erroneous ones with probability $1 - p$.

The PBS can be traced back to Horstein (Horstein, 1963). In PBS a probability distribution is mapped over the search space and is gradually updated using a Bayesian methodology under the assumption that the environment noise p is known a priori. The search space is then continuously explored using the median of the posterior distribution as the point of interest. It has been shown that PBS has a geometric rate of convergence under the latter assumptions (Waeber et al., 2013).

As the noise p is assumed to be given, one can simply invoke (7) to calculate the posterior distribution:

$$P(d \mid Q) \propto P(Q \mid d)\, P(d). \tag{7}$$

Here, $P(Q \mid d)$ is the conditional probability of obtaining answer Q (point to the right). That is, for every location d to the left of X^*, the probability that the oracle directs the decision maker to the right is $P(Q \mid d) = p$. And conversely, $P(Q \mid d) = 1 - p$ for d to the right of X^*.

To explicitly represent PBS' dependence on knowing p beforehand, we can cast (7) in terms of (5) and (6). The resulting model is identical to TS-SPL, with the major difference that PBS employs the median to explore the search space. We denote this new and improved scheme PBS-M.

We also observe that due to its simple nature, PBS is particularly well-suited for parallel computing environments (Pallone et al., 2014), as opposed to more traditional stochastic approximation methods (Kushner and Yin, 1987).

PowerTest-Probabilistic Bisection Search

In a recent paper, Frazier et al. (Frazier et al., 2019) demonstrated an alternative approach to removing the dependency of PBS on knowing the fixed noise probability p. Instead of

[2] In this context this scheme also covers the Stochastic *Binary* Search

applying a Bayesian prior over p, as done in TS-SPL, they introduce a frequency-based approach, referred to as PowerTest-PBS (PT-PBS). PT-PBS is based on repeatedly sampling the underlying function $g(x)$ until a pre-specified confidence α is archived on a hypothesis test over the sign of the feedback of $g(x)$. They further demonstrated that the asymptotic convergence of PT-PBS is similar to that of Stochastic Approximation (SA).

Generalized Binary Search

The Generalized Binary Search (GBS) problem can be formulated as follows (Nowak, 2008, 2011). Consider a collection of unique binary-valued functions H defined on a domain X. Each $h \in H$ is defined as a mapping from X to $\{-1, 1\}$. Assume that there exists an optimal function $h^* \in H$ that produces the correct binary labeling for each $x \in X$. For each query $x \in X$, the value of $h^*(x)$ is observed, possibly corrupted by independent binary noise. The objective is then to determine the function h^* using as few queries as possible. In this paper, we restrict H to the class of threshold binary functions with the effect of turning the GBS into an informative N-door puzzle.

If the feedback is noiseless then the problem simplifies to the combinatorial problem of finding an optimal decision tree in the H space, a problem that Hyafil and Rivest showed to be NP-complete (Hyafil and Rivest, 1976; Nowak, 2011).

The Soft-Decision Generalized Binary Search (SDGB-Search) (Nowak, 2008, 2011) is the *state-of-art* algorithm for finding $h^*(x) \in H$ when the probability of binary noise is less than 1/2, that is, for informative environments.

Similar to TS-SPL, SDBG-search employs a probabilistic model that for time step n assigns a probability $p_n(h)$ to each $h \in H$. However, for each time-step, it decides which $x \in X$ is queried next based on a deterministic heuristic:

$$\arg\min_{x \in X} \sum_{h \in H} |p(h)h(x)| \tag{8}$$

SDGB uses the following equation to determine and update $p_n(h)$ at each time step:

$$p_{i+1}(h) \propto p_i(h)\beta^{(1-z_i(h))/2}(1-\beta)^{(1+z_i(h))/2}. \tag{9}$$

Here, $z_i(h) = h(x_i)y_i$ and $y_i \in \{-1, 1\}$ are the responses from $h^*(x_i)$. Simplifying (9), we observe that $z_i(h)$ represents an *AND* operator that takes on the value 1 if $h(x_i)$ is equal to $h^*(x_i)$ and -1 otherwise. Furthermore, we note that since $z_h(i) \in \{-1, 1\}$, then one of $1 - z_i(h)$ and $1 + z_i(h)$ will have to take the value 2, while the other takes the value 0.

By applying the transformation $\pi = 1 - \beta$, we can rewrite (9) as:

$$p_{i+1}(h) \propto = \begin{cases} p_i(h) \times \pi & \text{if } y_i = h^*(x_i) \\ p_i(h) \times (1-\pi) & \text{else} \end{cases} \tag{10}$$

.

This update scheme is identical to the one in PBS and thus suffers from the same limitation (noise probability is assumed to be known a priori). In the same manner as we enhanced PBS by employing our Bayesian TS-SPL scheme, we can make SDGB parameter-free using (5,6). In the following, we will denote this improved version of SDGB as SDGB-M.

Burnashev-Zigangirov Algorithm

The Burnashev-Zigangirov (BZ) algorithm (Burnashev and Zigangirov, 1974) is one of the most widely used algorithms for solving the discrete PBS problem and has in particular been used in active learning (Singh et al., 2006; Castro and Nowak, 2006). The BZ algorithm searches for a point θ^* that is located on a line. This line is discretized into m bins and we are only allowed to query the borders of the bins for the direction of θ^*. The BZ algorithm suffers from the same practical limitation as PBS and SDGB, namely a dependency on knowing the exact noise level.

We will now show how the BZ algorithm can be improved in a similar fashion as PBS and SDGB, leveraging our Bayesian model. Let $a_i(j)$ denote the probability of θ^* residing in bin I_i at time-step j. The probability mass function (pmf) of all the bins is therefore $\mathbf{a}(j) = \{a_1(j), a_2(j), \ldots, a_m(j)\}$ with its cumulative density function (cdf) denoted as $\mathbf{A}(j)$.

To decide which point to investigate next (i.e., decide a value for X_{j+1}), the BZ algorithm selects one of the two closest points to the median of $\mathbf{a}(j)$. We denote this point $k = k(j+1)$. The binary response variable $Y_{j+1} = \mathbb{1}\{X_{(j+1)} \geq \theta^*\}$ is observed with probability $1-\alpha$, whereas $Y_{j+1} = \mathbb{1}\{X_{(j+1)} < \theta^*\}$ is observed with probability α (the noise probability).

To update the probability distribution over $\mathbf{a}(j)$, we introduce $\beta = 1-\alpha$ and $\tau = 2A(k(j+1)) - 1$. For $i \leq k$, we then have

$$a_i(j+1) = a_i(j) \begin{cases} \frac{2\alpha}{1-\tau(\beta-\alpha)} & \text{if } Y_{j+1} = 0 \\ \frac{2\beta}{1+\tau(\beta-\alpha)} & \text{if } Y_{j+1} = 1 \end{cases}$$

and for $i > k$ we have:

$$a_i(j+1) = a_i(j) \begin{cases} \frac{2\beta}{1-\tau(\beta-\alpha)} & \text{if } Y_{j+1} = 0 \\ \frac{2\alpha}{1+\tau(\beta-\alpha)} & \text{if } Y_{j+1} = 1 \end{cases}$$

To make the BZ algorithm parameter-free, we first note that for any given noise $t \in T$, we have that $\beta = t$, $\alpha = 1-t$, $\beta - \alpha = 2t-1$, and $\tau = A_k(j) - (1 - A_k(j))$. After some simple algebraic manipulations, it turns out that the updating scheme of the BZ algorithm is identical to PBS except that:

1. The BZ algorithm calculates the normalizing factor as a part of the updating rule instead of using the likelihood value, and then later normalizes as PBS does.

2. The BZ algorithm samples on the interval edges while PBS samples the midpoints of each interval.

To obtain an enhanced parameter-free version of the BZ algorithm, we simply replace α as a pre-determined constant with a prior distribution that we marginalize out using (5) and (6). We denote the resulting scheme BZ-M.

3. Empirical Results

In this section, we evaluate the performance of TS-SPL empirically, in comparison with competing schemes. We investigate both the effect that the various parameter settings have on the behavior, as well as the capability of TS-SPL to handle different applications, including SPL and SRF problems. Unless otherwise noted, the empirical results report the average of 10 000 independent trials.

For some of the applications we investigate here, we do not find any existing scheme that handles deceptive environments. Instead, the schemes we identified assume that feedback is on average informative. To make the comparison fair, we thus introduce TS-SPL-INF, which is configured with the precondition that the feedback is informative. This modification also serves to exemplify one of the advantages of our Bayesian approach — we can tailor the the prior distribution of the noisy probability for the task at hand. Note that this informed prior is equivalent to the priors we use for the other probability theory based schemes we introduced in this paper, namely, PBS-M and SDGB-M.

Further note that we apply a fixed set of parameter values across the whole suite of experiments, set to optimize overall performance. For SHSSL (Zhang et al., 2016) and HSSL (Yazidi et al., 2012a) we used a tree branching factor of $D = 8$, and for ASS (Tao et al., 2013) we set $N_{\max} = 256$ and $N_{\min} = 1$. For OCBA (Zhang et al., 2016), we set $n_0 = 15$ and $\theta = 1/256$. We additionally set the confidence γ of PT-PBS (Frazier et al., 2019) to 0.55 based on a comprehensive brute force search for the best value. The prior used for TS-SPL is uniform over the unit interval and is discretized as $|D| = 201$ and $|T| = 101$. For the informative schemes TS-SPL-INF, PGA-M, SGDB-M, BZ-M, we use the same prior for the doors as for TS-SPL, $|D| = 201$ however, we use a uniform prior over the interval $(0.5, 1]$ for truthfulness, with $|T| = 51$.

We will in the following subsections investigate (1) the effect of different priors on TS-SPL; (2) TS-SPL's ability to identify the nature of the underlying stochastic environment; (3) the ability to solve the SPL problem; and (4) performance on SRF problems – a particularly intriguing class of deceptive environments that arises naturally as a result of the properties of stochastic root finding.

3.1. Sensitivity to Discretization and Distribution of Prior

Although TS-SPL is a parameter free scheme, it depends on defining $D \times T$, the set of all possible N-Door Puzzles, and then formulating a prior distribution over this space. We here investigate to what degree the performance of TS-SPL is affected by the degree of discretization, that is, the cardinality of $D \times T$.

To measure performance, we count how many time steps passes before 95% of the probability mass is contained within the target interval I, that is, $P(I|\text{Observed History}) \geq 0.95$. We refer to this event as convergence of the learning process.

From Table 2, we observe that the cardinality of D, in fact, does affect the performance of TS-SPL. As $|D|$ increases, so does the time it takes before TS-SPL converges. However, it is evident that the relationship between convergence time and $|D|$ is non-linear. Indeed, the increase in convergence time is insignificant even when doubling the number of doors from 3200 to 6400. The behaviour reported in the table thus indicates a logarithmic relation between $|D|$ and convergence time.

To see how the cardinality $|T|$ of T affects performance, we increase $|T|$ from 50 to 3200, fixing $|D|$ to 100. From Table 3, it is clear that $|T|$ does not significantly affect performance.

$\|D\|$:	100	200	400	800	1600	3200	6400
Convergence Steps:	31.4	36.0	38.9	39.4	39.3	40.2	40.9

Table 2: Convergence steps for TS-SPL solving the N-Door Puzzle with $\lambda^* = 0.15$, $I = \{0.15 \pm 0.01\}$, $T = \{0.8\}$, and $\pi = 0.8$.

$\|T\|$:	50	100	200	400	800	1600	3200
Convergence Steps:	51.6	50.8	48.4	52.1	51.0	52.4	52.1

Table 3: Convergence steps for TS-SPL solving the N-Door Puzzle with $\lambda^* = 0.15$, $I = \{0.15 \pm 0.01\}$, $|D| = 101$, and $\pi = 0.8$.

Another advantage of our Bayesian scheme is the ability to incorporate prior information to guide the algorithm. On the other hand, specifying an incorrect prior can potentially deteriorate performance instead of enhancing it. In Table 4, we provide performance results from employing an informed prior over T and D. With the correct underlying values $\lambda^* = \pi = 0.85$, we specify three types of priors: Correct $\propto N(\mu = 0.85, \sigma = 0.3)$, Incorrect $\propto N(\mu = 0.15, \sigma = 0.3)$ and Flat (all solutions equally probable), denoted C, I, and F, respectively. We can see the effect of these different priors in Table 4. In brief, having a correct prior over the doors contributes more to convergence time than having a correct prior over the truthfulness of the guards. The disadvantage of setting an incorrectly biased prior is also evident, as the flat prior performs better than any combination involving a incorrectly biased prior.

3.2. Tracking the Truthfulness of the Environment

An interesting property of TS-SPL is its ability to provide a distribution over the truthfulness π of the environment. This can be a significant advantage because information on π can be leveraged in various ways. As an example, information on π can be used in the case of repeated trials, where the information from previous trials can be used as a prior in subsequent trials, greatly increasing convergence speed (cf. Section 3.1). Figure 1 plots the probability of each level of noise as the TS-SPL progresses with noise probability $\pi = 0.15$ (a highly deceptive environment). As seen, TS-SPL is capable of quickly estimating π accurately.

3.3. Stochastic Point Location

The N-Door Puzzle, as outlined in the introduction, is dependent on two variables λ^* and π^*, with λ^* specifying the door leading to freedom and π^* the truthfulness of the guards. Since the N-Door Puzzle does not pose any spatial requirements on the placements of the doors we can generate a mapping from the N-Door Puzzle to the SPL problem by uniformly placing the doors over the unit interval.

We here use so-called regret to measure performance because not all of the schemes evaluated in this section are Bayesian. Regret is further typical for evaluating multi-armed bandit algorithms Regret can be stated as the cumulative penalty from selecting sub-optimal

Door	Truthfulness	Convergence
F	F	36.4
F	C	35.7
F	I	41.2
C	F	30.2
C	C	30.0
C	I	40.5
I	F	46.4
I	C	45.2
I	I	113.1

Table 4: Convergence steps for TS-SPL solving the N-Door Puzzle with different priors: C - Correct Prior, F - Flat Prior, I - Incorrect prior. Here, $\lambda^* = 0.85$, $I = \{0.15 \pm 0.01\}$, $|T| = |D| = 101$, and $\pi = 0.85$.

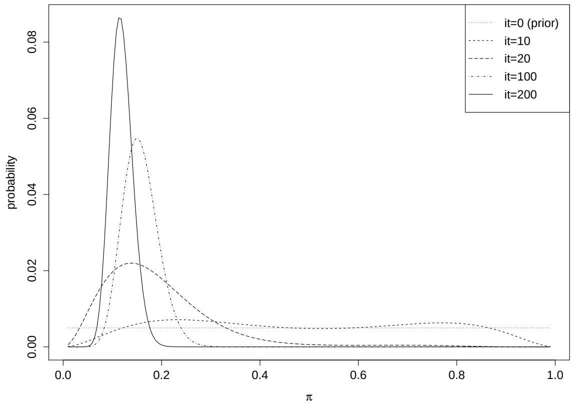

Figure 1: TS-SPL maintains a posterior distribution over π. Here, the true underlying value of π is 0.15. The figure shows the posterior distribution of π after various number of iterations during a single run of TS-SPL. As seen, TS-SPL obtains a sharply peaked posterior over π after only 20 iterations.

actions. In the case of SPL we define regret as the (unsigned) distance between the selected point x and the optimal point λ^*.

3.3.1. Informative SPL

We first evaluate the performance of TS-SPL and TS-SPL-INF on an informative SPL problem, in comparison with algorithms designed to handle informative environments. To the best of our knowledge this is the first time both the family of PBS based schemes and the family of SPL based schemes are compared.

The performance of the different schemes is summarized in Table 5. One significant observation is the performance difference between the Learning Automata (LA) based schemes (HSSL and SHSSL) and the Bayesian schemes. It is clear that performance-wise, Bayesian schemes significantly outperform the LA based schemes. However, it should be noted that the LA based schemes require less memory and run faster than the Bayesian ones due to their simplicity.

As seen in Table 5, the distance $|\frac{1}{2} - \lambda^*|$ is an important metric for how hard a particular SPL problem is to solve. This can be explained by the fact that most schemes start exploring from the center. Thus, if λ^* is far from the center, such a scheme needs more evidence before it starts exploring the peripheral regions of the search space. This is particularly apparent for PBS-M as its performance peaks in the case where $\lambda^* = 0.25$, even when faced with significant noise ($\pi = 0.65$).

Since PBS-M pursues the median of the probability distribution, we can say that PBS-M is conservative in its exploration. This is because it takes significant more evidence to move the point of exploration compared to TS-SPL. TS-SPL, on the other hand, has a tendency to explore too much. Indeed, as noted by Lattimore (Lattimore, 2015), using TS for exploration can lead to over-exploration when facing high variance distributions. In the low noise scenarios, however, NGBS-M is the most efficient scheme, exploring deterministically. For PT-PBS, we observe throughout all the experiments significantly worse performance than PBS-M. We thus omit the PT-PBS results for the remaining experiments, focusing on PBS-M instead.

Finally, from Table 6 we observe that TS-SPL-INF exhibits the lowest standard deviation overall, and is consequentially the scheme that consistently perform closest to its expected regret for every trial. This is in sharp contrast to PBS-M who outperform TS-SPL when it comes to average regret, but is unable to do so consistently. NGBS-M also displays significant variance in high noise scenarios.

3.4. Stochastic Point Location in Deceptive Environments

With the underlying π taking on values in the interval $[0, 1]$, we test TS-SPL, CPL-AdS (Oommen et al., 2003) and SHSSL (Zhang et al., 2016) for speed of convergence as well as accumulation of regret. However, since CPL-AdS operates in a two-phase manner, direct comparison with TS-SPL and SHSSL is inappropriate (the latter schemes also operate online). Oommen et al. states that this decision phase needs approximately 200 time steps (Oommen et al., 2003), and by this time TS-SPL is already close to converging to the actual solution. Table 7 further explores this difference. As seen, TS-SPL outperforms CPL-AdS by several orders of magnitude, also outperforming SHSSL.

	Avg Regret $\lambda^* = 0.25$	Avg Regret $\lambda^* = 0.85$	Avg Regret $\lambda^* = 0.95$
TS-SPL	29.2 / 9.8 / 5.1	36.4 / 12.9 / 6.2	57.3 / 20.3 / 10.1
TS-SPL-INF	22.2 / 7.3 / 3.7	**22.5** / 7.7 / 3.8	**23.9** / 8.7 / 4.3
PBS-M	**9.8** / 4.0 / 2.6	32.7 / 14.2 / 8.5	52.1 / 29.6 / 16.9
PT-PBS	245.4 / 247.5 / 247.4	548.2 / 751.5 / 785.2	551.4 / 815.0 / 828.4
BZ-M	23.5 / 5.9 / 2.2	27.5 / 6.3 / 2.5	35.1 / 9.6 / 3.4
NGBS-M	36.9 / **3.5** / **1.0**	48.9 / **4.5** / **1.5**	68.5 / **7.1** / **2.3**
ASS	45.8 / 17.0 / 6.7	30.4 / 8.9 / 3.6	38.8 / 11.7 / 3.9
OCBA	70.8 / 47.4 / 35.2	89.9 / 55.8 / 37.1	112.1 / 78.4 / 48.8
HSSL	117.3 / 23.1 / 8.2	111.7 / 16.7 / 4.8	131.5 / 19.1 / 5.3
SHSSL	152.2 / 32.6 / 11.8	151.8 / 23.5 / 6.5	175.1 / 26.1 / 7.3

Table 5: Average regret for the different schemes in an informative SPL. The result is reported in the format $a/b/c$, where a is the average regret for $\pi = 0.65$, b for $\pi = 0.75$, and c for $\pi = 0.85$. The number of time steps per trial is 1000.

	Std. dev. $\lambda^* = 0.25$	Std. dev. $\lambda^* = 0.85$	Std. dev. $\lambda^* = 0.95$
TS-SPL	16.8 / 5.9 / 2.6	20.5 / 6.5 / 3.1	30.9 / 10.3 / 4.6
TS-SPL-INF	**13.8** / **4.2** / 2.0	**14.2** / **4.4** / 2.5	**15.7** / **5.7** / 2.4
PBS-M	15.2 / 10.3 / 10.2	69.1 / 40.9 / 31.5	94.2 / 71.1 / 56.6
PT-PBS	20.1 / 19.0 / 13.6	145.6 / 31.0 / 22.8	192.1 / 60.5 / 55.8
BZ-M	30.4 / 8.9 / 3.1	40.8 / 9.8 / 4.9	48.8 / 15.3 / 5.3
NGBS-M	68.5 / 8.9 / **0.9**	83.7 / 13.6 / **1.4**	108.7 / 19.4 / **1.6**
ASS	51.6 / 22.4 / 10.1	47.8 / 15.7 / 5.4	62.3 / 23.1 / 4.6
OCBA	46.2 / 27.6 / 19.4	63.9 / 43.9 / 25.6	76.1 / 64.6 / 41.9
HSSL	71.7 / 16.1 / 4.6	83.6 / 16.1 / 4.2	94.8 / 19.4 / 4.5
SHSSL	89.7 / 23.5 / 6.2	108.5 / 23.4 / 5.8	126.5 / 27.7 / 6.4

Table 6: Standard deviation for the different schemes in an informative SPL. The result is reported in the format $a/b/c$, where a is the standard deviation for $\pi = 0.65$, b for $\pi = 0.75$, and c for $\pi = 0.85$. The number of time steps per trial is 1000.

	$\pi = 0.85$	$\pi = 0.15$
TS-SPL ($\lambda^* = 0.85$)	6.2	**6.2**
CPL-AdS ($\lambda^* = 0.85$)	501.6 / 354.9	842.8/502.3
PBS-M ($\lambda^* = 0.85$)	31.5	77.5
BZ-M ($\lambda^* = 0.85$)	4.9	352.5
NGBS-M ($\lambda^* = 0.85$)	**1.4**	191.2
SHSSL ($\lambda^* = 0.85$)	6.5	6.5

Table 7: Cumulative regret for the deceptive SPL problem after $N = 1000$ time steps. For CPL-AdS, we report both the total accumulated regret, as well as regret obtained after the nature of the environment has been decided.

Another interesting observation is that the performance of TS-SPL is symmetric around 0.5. Further note that SHSSL fails to converge for $\pi \in [0.382, 0.682]$, as stated earlier. Hence, SHSSL is effectively operating with a 30% smaller search space for π than both TS-SPL and CPL-AdS.

After modifying PBS, NGBS and BZ to support a Bayesian model of truthfulness, we can use the same prior that we apply in TS-SPL also for these schemes, leading to PBS-M, NGBS-M and BZ-M. The effect of this enhancement to existing schemes is summarized in Table 7. As clearly seen, the query selection method for these schemes is not suited to handle deceptive environments.

3.5. Stochastic Root-Finding

The deterministic root finding problem concerns locating a root x^* of a function $g(x)$, defined over an interval (a, b) [i.e., finding x^*, $g(x^*) = 0$]. We assume that $g(x)$ is unknown, however, an oracle returns the value of $g(x)$ at any point x queried. The problem is then how to determine the root x^* using as few queries as possible. One approach to solving the deterministic root finding problem is the Bisection Method. This approach halves the search space in each iteration by continually querying the oracle using the midpoint of the remaining search space.

If the response from the oracle is noisy, we obtain the SRF problem (Pasupathy and Kim, 2011). We define the SRF problem as follows. For any $x \in (0, 1)$, the oracle generates a sample $Y(x) = g(x) + w_{\text{noise}}$, where w_{noise} is a random variable with mean zero. Let $S(x)$ denote the sign of $Y(x)$: $S(x) = \text{sgn}[Y(x)]$. Notice that the noise w_{noise} may render $\text{sgn}[Y(x)]$ different from $\text{sgn}[g(x)]$. Thus, with noisy feedback, the Bisection Method may discard the wrong half of the search space. The challenge is then how to select a sequence of queries $x_1, x_2, \ldots, x_n$ to gather information on x^*, so that the final query x_n is close to x^*, $|x_n - x^*| < \epsilon$, despite the noise (Waeber et al., 2011). Note that in the SRF problems we investigate here, $S(x)$ returns $\text{sgn}[g(x)]$ with probability π and $-\text{sgn}[g(x)]$ with probability $1 - \pi$.

The traditional approach to solving SRF problems is to apply a variant of Stochastic Approximation (SA) (Robbins and Monro, 1951; Kiefer and Wolfowitz, 1952). Implementationwise SA methods[3] extend or modify the iterative Newton-Raphson algorithm to handle noise:

$$x_{n+1} = x_n - a_n Y_n(x_n)$$

where $\{a_n\}$ is a sequence of step lengths that decreases as n increases.

Approaches for applying SA to the SRF problem has been extensively studied in the literature. It is outside the scope of this article to give a full literature review, however, interested readers are referred to surveys in recent studies (Lai, 2003; Asmussen and Glynn, 2007; Pasupathy and Kim, 2011). As there exists a myriad of different SA approaches, we have selected one of the more fundamental ones to form a basis for contrasting the different schemes.

Note that the SA approach we use here requires that $g(x)$ is monotonic. This can be explained as follows. The main difference between SRF and SPL is that, unlike SPL, the SRF oracle does not directly provide feedback on the direction of the root x^* from the query location x. However, for monotonic $g(x)$, one can obtain this direction from the sign of $Y(x)$, $S(x) \in \{-1, 1\}$ and from the derivative $g'(x)$ of $g(x)$. If $g(x)$ is increasing and $S(x) = 1$, then the direction derived from the oracle is "to the left of x" [and "to the right" if $S(x) = -1$]. Conversely, if $g(x)$ is decreasing and $S(x) = 1$, then the direction obtained is "to the right of x" [and "to the left" if $S(x) = -1$].

TS-SPL, on the other hand, does not need to know the derivative of $g(x)$. Indeed, $g(x)$ does not even need to be monotonic. Instead, TS-SPL merely requires an arbitrary mapping of the sign $S(x)$ to a direction. One could, for instance, define positive to mean "left", $S(x) = 1 \Rightarrow$ *left*, and negative to mean "right", $S(x) = -1 \Rightarrow$ *right*. If it turns out that the opposite is the case, TS-SPL will recognize that the feedback is deceptive and still solve the problem. An informative scheme, on the other hand, will be misled in such a deceptive environment.

Informative SPL schemes can also be used for SRF problems. However, then we need an initial sampling step that decides the nature of the function $g(x)$. Learning whether the function $g(x)$ starts above zero and falls below zero, or vice versa, can be done by repeatedly querying a single point on the edge of the interval $(0, 1)$, obtaining multiple samples from either $S(x)$. In brief, by estimating $E[S(x)]$, we can decide the nature of $g(x)$. To gain insight into how many repeated samples are sufficient for estimating $E[S(x)]$ accurately, we employ the two sided Hoeffding's inequality $P(|\bar{X} - E[\bar{X}]| \geq \delta) \leq 2e^{-2n\delta^2}$. Here, $\bar{X}$ is the average of n queries at x and δ is a value such that $|\pi - \frac{1}{2}| \geq \delta$. Setting the rhs. equal to p and solving for n, we obtain $n \geq -\frac{\log(p/2)}{2\delta^2}$. Plugging in for $\delta = 0.05$ and $p = 0.99$ we obtain $\lceil n \rceil = 62$. Thus we are 99% sure of our estimate of $S(x)$, given that $|\pi - 0.5| \geq 0.05$.

The functions that we use to measure performance and compare schemes are illustrated in Figure 2. From Table 8, 9 and 10 it is clear that TS-SPL is the most efficient root solver among state-of-the-art schemes. We believe this largely comes from the fact that it simultaneously learns the nature of the oracle (informative or deceptive), as well as trying to locate the root x^*. In addition, there is the risk that the sampling procedure that the other schemes apply to determine which direction $g(x)$ is increasing may conclude with the wrong answer. If this happens, none of the schemes depending on the sampling will

[3] The form of SA shown here is also referred to as Classical Stochastic Approximation (CSA) as it closely resembles the original form proposed by Robbins and Monro (Pasupathy and Schmeiser, 2010).

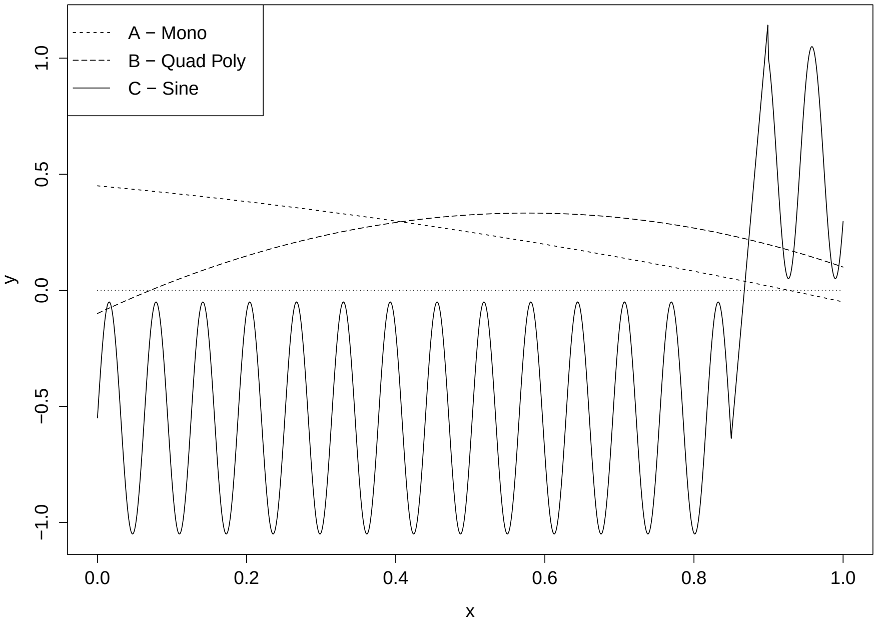

Figure 2: The three functions A, B and C for benchmarking stochastic root finding schemes.

Func A	Avg Regret. $\pi = 0.65$	Avg Regret. $\pi = 0.75$	Avg Regret $\pi = 0.85$
TS-SPL	46.0	17.1	8.6
TS-SPL-INF	55.2 (24.3)	39.1 (8.1)	35.0 (4.1)
PBS-M	55.1 (24.2)	40.3 (9.3)	35.2 (4.2)
NGBS-M	53.4 (22.4)	35.3 (4.3)	32.9 (1.9)
BZ-M	63.8 (32.8)	39.9 (8.9)	33.8 (2.8)
SA	**32.4**	**14.8**	**5.4**
ASS	80.4 (49.4)	38.7 (7.7)	33.5 (2.5)
HSSL	60.4 (29.4)	45.5 (14.6)	35.8 (4.8)
SHSSL	62.8	20.2	6.7
CPL-AdS	162.1 (107.9)	146.3 (97.4)	135.3 (90.1)

Table 8: Average residuals for the different schemes when finding the root of the monotonic function A under various noise levels. The root is $x^* = 0.07104$. The results are given in the format ”average residuals (average residuals after sampling)” for each scheme. For CPL-AdS the sampling period is the estimation period (epoch 0) as defined by the scheme. The number of iterations per trial is 250.

Func B	Avg Regret. $\pi = 0.65$	Avg Regret. $\pi = 0.75$	Avg Regret $\pi = 0.85$
TS-SPL	47.1	**17.8**	**8.5**
TS-SPL-INF	53.8 (22.9)	39.5 (8.5)	35.1 (4.1)
PBS-M	**41.1** (10.2)	35.3 (4.31)	33.3 (2.3)
SGBS-M	50.6 (19.7)	35.7 (4.7)	33.0 (2.0)
BZ-M	60.3 (29.4)	39.6 (8.7)	33.6 (2.6)
SA	175.1	204.5	223.3
ASS	81.6 (50.6)	39.5 (8.7)	40.2 (9.0)
HSSL	85.3 (54.4)	50.6 (19.6)	39.0 (8.0)
SHSSL	75.4	30.8	12.7
CPL-AdS	117.9 (109.3)	116.7 (107.1)	144.9 (96.5)

Table 9: Average residuals for the different schemes when finding the root of the quadric function B under various noise levels. The root is $x^* = 0.9270$. The results are given in the format ”average residuals (average residuals after sampling)” for each scheme. For CPL-AdS the sampling period is the estimation period (epoch 0) as defined by the scheme. The number of iterations per trial is 250.

Func C	Avg Regret. $\pi = 0.65$	Avg Regret. $\pi = 0.75$	Avg Regret $\pi = 0.85$
TS-SPL	**36.9**	**13.7**	**6.4**
TS-SPL-INF	52.8 (21.9)	38.8 (7.8)	34.9 (3.9)
PBS-M	49.6 (18.7)	39.2 (8.3)	34.3 (3.3)
SGBS-M	47.2 (16.2)	34.6 (3.6)	32.5 (1.6)
BZ-M	58.8 (27.9)	38.4 (7.4)	33.7 (2.7)
SA	149.0	178.0	185.0
ASS	54.3 (23.4)	39 (8.0)	33.5 (2.5)
HSSL	75.2 (44.3)	44.3 (13.4)	35.6 (4.6)
SHSSL	56.5	18.4	**6.4**
CPL-AdS	153.0 (101.6)	156.2 (103.7)	165.0 (109.6)

Table 10: Average residuals for the different schemes when finding the root of the sinusoidal function C under various noise levels. The root is $x^* = 0.8675$. The results are given in the format ”average residuals (average residuals after sampling)” for each scheme. For CPL-AdS the sampling period is the estimation period (epoch 0) as defined by the scheme. The number of iterations per trial is 250.

converge towards the root x^*. A perhaps even stronger advantage of TS-SPL is that it can be applied to a wide range of functions without regards to the presence of local extrema. SA, on the other, only performs well for monotonic functions as exemplified in Table 8.

4. Conclusions and Further Work

In this paper, we investigated a novel reinforcement learning problem derived from the so-called "N-Door Puzzle". This puzzle has the fascinating property that it involves stochastic *compulsive liars*. Feedback is erroneous on average, systematically misleading the decision maker. This renders traditional reinforcement learning (RL) based approaches ineffective due to their dependency on *"on average"* correct feedback.

To solve the problem of deceptive feedback, we recast the problem as a challenging variant of the multi-armed bandit problem, referred to as the *Stochastic Point Location* (SPL) problem. In SPL, the decision maker is only told whether the optimal point on a line lies to the "left" or to the "right" of a current guess, with the feedback being erroneous with probability $1 - \pi$. Solving this problem opens up for optimization in continuous action spaces with both *informative* and *deceptive* feedback.

Our solution to the above problem, introduced in the present paper, is based on a novel Bayesian representation of the solution space that is both compact and scalable. This model simultaneously captures both the location of the optimal point, as well as the probability of receiving correct feedback π. We further introduced an accompanying Thompson Sampling (TS) guided Stochastic Point Location (TS-SPL) scheme for balancing exploration against exploitation. By learning π, TS-SPL supports deceptive environments that are lying about the direction of the optimal point.

We used TS-SPL to solve the Stochastic Point Location (SPL) problem and outperformed all of the Learning Automata driven methods. However, by enhancing the Soft Generalized Binary Search (SGBS) scheme with our Bayesian representation of the solution space, SGBS was able to outperform TS-SPL under informative feedback. For deceptive SPL problems, TS-SPL outperformed all of the existing state-of-art schemes by several orders of magnitude, even when the latter schemes were supported by our Bayesian model.

We also applied TS-SPL to the *Stochastic Root Finding* (SRF) problem. We further demonstrated that SRF can be seen as a deceptive problem, allowing TS-SPL to outperform existing dedicated state-of-art SRF schemes by an order of magnitude. Thus, TS-SPL can be considered state-of-the-art for both deceptive SPL and for SRF, while yielding comparable results to the top performing schemes in the case of informative SPLs.

Despite the above performance gains, TS-SPL is based on Thompson Sampling, which is known to have a tendency to over-explore high variance reward distributions (Lattimore, 2015). In future work, it is therefore interesting to investigate mechanisms that eliminate or reduce this tendency, to further increase convergence speed.

Another important avenue for future work is the establishment of theoretical results, including proof of convergence, to corroborate the purely empirical findings presented in this paper. We suggest that a promising starting point for such an endeavour would be to combine the theoretical properties of TS (Agrawal and Goyal, 2012, 2013a; Dong and Van Roy, 2018) with the theoretical results of PBS (Waeber et al., 2013; Frazier et al., 2019), as they are closely related.

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