Journal of Machine Learning Research 27 (2026) 1-49 Submitted 4/25; Revised 3/26; Published 6/26

Nonparametric Spectral Density Estimation using Interactive Mechanisms under Local Differential Privacy

Cristina Butucea CRISTINA.BUTUCEA@ENSAE.FR
CREST, ENSAE
Institut Polytechnique de Paris
Palaiseau, 91120, France

Karolina Klockmann KAROLINA.KLOCKMANN@UNIVIE.AC.AT
Department of Statistics and Operations Research
Universität Wien
Wien, 1090, Austria

Tatyana Krivobokova TATYANA.KRIVOBOKOVA@UNIVIE.AC.AT
Department of Statistics and Operations Research
Universität Wien
Wien, 1090, Austria

Editor: Weijie Su

Abstract

We study the problem of estimating the spectral density of a centered stationary Gaussian time series under local differential privacy constraints. Specifically, we propose new interactive privacy mechanisms for three tasks: recovering a single covariance coefficient, recovering the spectral density at a fixed frequency, and global recovery. Our approach achieves faster rates through a two-stage process: we first apply the Laplace mechanism to the truncated value, and then use the resulting privatized sample to learn about the dependence mechanism in the time series. For spectral densities belonging to Hölder and Sobolev smoothness classes, we demonstrate that our algorithms improve upon the non-interactive mechanism of Kroll (2024) for small privacy parameter α, since the pointwise rates depend on $n\alpha^2$ instead of $n\alpha^4$. Moreover, we show that the rate $(n\alpha^4)^{-1}$ is optimal for estimating a covariance coefficient with non-interactive mechanisms. However, the L_2 rate of our interactive estimator is slower than the pointwise rate. We show how to use these procedures to provide a bona fide locally differentially private estimator of the entire covariance matrix. A simulation study validates our findings.

Keywords: autocovariance function, non-interactive privacy mechanisms, sequentially-interactive privacy mechanism

1. Introduction

In recent years, data privacy has become a fundamental problem in statistical data analysis. As the collection, storage, and analysis of personal data increase, the challenge lies in publishing accurate statistical results while protecting the privacy of individuals whose data is used. At the same time, traditional privacy techniques, such as anonymization, are becoming increasingly ineffective due to the growing availability of publicly accessible information, which facilitates the re-identification of individuals by linking anonymized data with external data sets. To address these concerns, differential privacy has been proposed as a rigorous mathematical framework that offers strong privacy guarantees. First introduced by Dwork (2006), differential privacy is typically classified into two main frameworks: global (or central) differential privacy (DP) and local differential privacy (LDP).

Global differential privacy relies on a centralized curator who collects confidential data from n individuals and produces a privatized output derived from the complete data set. Only this privatized output is shared publicly, ensuring that modifying a single entry in the data set has minimal impact on the output's probability distribution. This makes it difficult for an adversary to determine whether a specific individual's data is included in the data set.

In contrast, local differential privacy eliminates the need for a trusted curator. Under LDP, each individual independently generates a privatized version of their data on their own device before sharing it. Consequently, only these privatized versions are available for analysis, ensuring that the original data remains private. While individuals maintain control over their original data, some level of interaction among the n individuals may be permitted. LDP offers stronger privacy guarantees than DP at the expense of lower data utility for analysis.

In this paper, we focus on local differential privacy to privatize a single sequence of n correlated observations $(X_1, ..., X_n) \in \mathbb{R}^n$. The ordering of the sequence may arise from the design of a study in which related individuals are surveyed sequentially, or from other factors, such as sensor networks where measurements are taken across different locations in a structured manner. Applications of this setting include snowball survey designs, in which individuals are interviewed sequentially, often introducing inherent correlations, whether spatial, social, or otherwise, among participants (Goodman, 1961). In such surveys, the variable of interest, X_i, might represent annual income or health-related attributes. In the following, we refer to $(X_1, ..., X_n)$ as an observation of a time series and to X_i as the data of the i-th individual, but the sequential structure may be caused by factors other than time. The proposed privacy mechanisms are also applicable for scenarios where the data $(X_1, ..., X_n)$ belongs to a single individual, and sharing only a privatized aggregated statistic is insufficient. For instance, $(X_1, ..., X_n)$ could represent an individual's energy consumption data. To optimize energy planning, providers may require detailed consumption patterns over time rather than just average usage. However, disclosing fine-grained consumption data could expose sensitive information, such as when a person is at home. This highlights the need to release a privatized version of the sequence $(X_1, ..., X_n)$ that preserves global characteristics, such as the correlation structure, while preventing inference of individualized behavioral patterns. Waheed et al. (2023) proposed a generalized random response mechanism to enable the secure sharing of energy consumption data. Specifically, depending on the outcome of a coin flip, either X_i is reported truthfully or replaced with a uniformly chosen value from $\{X_j\}_{j=1,...,i-1,i+1,...,n}$. This results in a new sequence that satisfies local differential privacy for an appropriately chosen coin-flip probability. Shanmugarasa et al. (2024) consider the privatization of a vector-valued time series representing the energy consumption of different household appliances. At each time instance, the energy consumption data is discretized to a set of energy levels and encoded as a binary vector, after which the randomized response mechanism of Warner (1965) is applied to ensure local differential privacy. While these works focus on algorithmic approaches, they lack a comprehensive analysis of how estimators of statistical quantities, such as the covariance function or the spectral density, are affected by the applied privacy mechanisms.

To formalize our setting, let $(X_t)_{t\in\mathbb{Z}}$ be a real-valued centered stationary Gaussian time series with (auto-)covariance function $\sigma : \mathbb{Z} \to \mathbb{R}$, $j \mapsto \sigma_j$, defined by $\sigma_j = \mathrm{Cov}(X_t, X_{t+j})$ with arbitrary $t \in \mathbb{Z}$. We assume that the sequence of covariances $(\sigma_j)_{j\in\mathbb{Z}}$ is absolutely summable. This guarantees the existence of the spectral density function f, which is then defined by

$$f(\omega) = \frac{1}{2\pi}\sum_{j\in\mathbb{Z}} \sigma_j \exp(-\mathtt{i}j\omega), \quad \omega \in [-\pi, \pi], \tag{1}$$

where $\mathtt{i}$ is the imaginary unit. In particular, f is a non-negative, 2π-periodic function which is symmetric at zero. Since the absolute summability of the sequence of covariances $(\sigma_j)_{j\in\mathbb{Z}}$ implies its square summability, the covariance function can be recovered from the spectral density function by the inverse Fourier transform.

We observe the sequence $(X_1, \ldots, X_n)$ drawn from the time series $(X_t)_{t\in\mathbb{Z}}$. In this paper, we are interested in the estimation of a single covariance coefficient σ_j for some fixed integer j, of the spectral density $f(\omega)$ at a fixed frequency $\omega \in [-\pi, \pi]$, and of the whole function f under local differential privacy constraints.

When designing locally differentially private mechanisms, two approaches can be distinguished: non-interactive and sequentially interactive mechanisms. In the non-interactive model, each individual independently generates a private representation Z_i of their original data X_i. In contrast, the sequentially interactive model allows the i-th individual to access previously privatized data $Z_1, ..., Z_{i-1}$, along with their own original data X_i to create their private output Z_i.

The framework of differential privacy was introduced for i.i.d. data, and fewer methods exist for the privatization of a sequence of correlated data. The literature on privatizing such data can be broadly categorized into transform-based, model-based, and nonparametric approaches. In the transform-based approach, the correlated data is transformed, for example, with a Fourier or wavelet transform, so that the data become approximately uncorrelated (Rastogi and Nath, 2010; Lyu et al., 2017; Leukam Lako et al., 2021). Then, existing privacy mechanisms can be applied, and the data is backtransformed. However, this is only possible in the central setting, when access to the whole original data is available. Another line of literature employs parametric models to describe data correlation within the framework of differential privacy, and designs mechanisms specific to this type of correlation. For instance, Zhang et al. (2022b) consider a Gaussian $AR(1)$ model, where a privatized version Z_i is generated as a convex combination of the true data X_i and the linear minimum mean squared error estimate $\hat{X}_i$. The estimate $\hat{X}_i$ is derived from Z_{i-1} and estimators for the mean, variance, and autocorrelation based on previously released data $Z_1, ..., Z_{i-1}$. By adding appropriately scaled, independent Gaussian noise, the mechanism achieves approximate local differential privacy. Cao et al. (2018) and Xiao and Xiong (2015) model temporal correlation with a Markov chain. Specifically, Xiao and Xiong (2015) construct a Hidden Markov Model from the noise-perturbed data sequence and release the sequence inferred from it. Finally, there is a nonparametric approach that imposes fewer restrictions on the type of correlation. To achieve global differential privacy, Wang and Xu (2017) proposed adding a sequence of correlated Laplace noise, with the correlation based on an estimate of the original sequence's autocorrelation. Kroll (2024) considers linear stationary processes with sub-Gaussian marginal distributions and assumes that the spectral density belongs to a certain smoothness class. Local differential privacy is ensured by truncating each data point X_i and adding independent Laplace noise. To our knowledge, Kroll (2024) is the only work that considers minimax optimal estimation of the spectral density under local differential privacy. For additional references on the privatization of sequential data, see the survey by Zhang et al. (2022a). Privatization of correlated, but non-sequential data, with multiple observations has been extensively studied in numerous works; see, for example, Zhu et al. (2014) and the references therein. In the context of covariance estimation, Amin et al. (2019) studied private estimation of the d-dimensional covariance matrix from a sample of n i.i.d. vectors $Y_1, ..., Y_n \in \mathbb{R}^d$, deriving error bounds under global differential privacy. Assuming a sub-Gaussian distribution for Y_i and entry-wise sparsity of the covariance matrix, Wang and Xu (2021) proposed an estimator for the covariance matrix from locally privatized data. Wang and Xu (2020) established the matching lower bounds, proving minimax optimality of this estimator. However, our setup corresponds to $n = 1$, in which case these approaches and their corresponding convergence rates differ from ours. Indeed, the cited papers privatize the entire vector $Y_i \in \mathbb{R}^d$ rather than its components. Our results show that coordinate-wise local differential privacy converges faster as the vector dimension increases under stationarity. Finally, alternative formulations of differential privacy that allow for correlated data, such as Bayesian differential privacy and the Pufferfish framework, have been introduced by Kifer and Machanavajjhala (2014) and Yang et al. (2015), respectively. McElroy et al. (2023) introduce the concept of Linear Incremental Privacy and propose an all-pass filtering method to protect a univariate stationary time series. This approach preserves the second-order structure of the time series while also accounting for potential auxiliary information the attacker may possess.

1.1 Our Contributions

We propose new sequentially interactive privacy mechanisms and estimators of a single covariance coefficient σ_j, of the spectral density at a fixed frequency ω: $f(\omega)$, and of the whole spectral density function f from a sample of size n generated from a centered Gaussian stationary time series. We assume that the spectral density function belongs either to the class of Sobolev smooth spectral densities $f \in W^{s,2}(L)$, $s > 1/2$, $L > 0$, or to the class of Hölder smooth spectral densities $f \in W^{s,\infty}(L_0, L)$, $s > 1/2$, L, $L_0 > 0$, both explicitly defined in Section 2. Previously, a non-interactive privacy mechanism for spectral density estimation under local differential privacy constraints was proposed by Kroll (2024). Their estimator $\hat{f}^{\mathrm{NI}}$ is shown to attain over the class $W^{s,2}(L)$ a convergence rate (up to log-factors) of

$$\mathbb{E}\|\hat{f}^{\mathrm{NI}} - f\|_2^2 \leq c(n\alpha^4)^{-\frac{2s}{2s+1}},$$

where $\alpha \in (0, 1)$ is the privacy parameter which gets smaller for higher privacy level and $c > 0$ is some constant depending on s, L. Their derived lower bounds are of the order $(n\alpha^2)^{-\frac{2s}{2s+1}}$, leaving open the minimax optimal convergence rate. For better comparability to our results, we first derive the convergence rate for σ_j and $f(\omega)$ with the non-interactive privacy mechanism of Kroll (2024), which, as summarized in Table 1, depend on $n\alpha^4$ as well.

Next, we propose sequentially interactive mechanisms, also called interactive mechanisms here, and estimators for a single Fourier coefficient σ_j and the spectral density $f(\omega)$ at a fixed frequency. For both, we show that our interactive mechanisms improve upon the currently known rates as α^4 is replaced by α^2. Specifically, we show the following bounds on the mean-squared error for $\alpha \in (0,1)$ and sufficiently large n (up to log-factors),

$$\begin{aligned}
\mathbb{E}|\hat{\sigma}_j - \sigma_j|^2 &\leq c\left(n\alpha^2\right)^{-1}, \quad &&\text{if } \sum_{k\in\mathbb{Z}} \sigma_k^2 \leq M_1, \\
\mathbb{E}|\hat{f}(\omega) - f(\omega)|^2 &\leq c_1 (n\alpha^2)^{-\frac{2s}{2s+1}}, \quad &&\text{if } f \in W^{s,\infty}(L_0, L), \\
\mathbb{E}|\hat{f}(\omega) - f(\omega)|^2 &\leq c_2 (n\alpha^2)^{-\frac{2s-1}{2s}}, \quad &&\text{if } f \in W^{s,2}(L),
\end{aligned}$$

where $c, c_1, c_2 > 0$ are constants depending only on $M_1 > 0$ and s, L, L_0, respectively.

It is important to note that these privacy mechanisms depend on the index j and, respectively, on the frequency ω, and thus cannot be directly generalized to estimate f globally. Employing the mechanism of Duchi et al. (2018) to privatize vectors in an ℓ_∞ ball, we propose a sequentially interactive privacy mechanism and an estimator for the spectral density function f and show the following bound on the mean integrated squared error, for sufficiently large n (up to log-factors),

$$\mathbb{E}\|\hat{f} - f\|_2^2 \leq c_3 (n\alpha^2)^{-\frac{2s}{2s+2}}, \quad \text{if } f \in W^{s,2}(L) \text{ or } f \in W^{s,\infty}(L_0, L),$$

where $c_3 > 0$ is a constant depending only on s, L, L_0.

This is the first setup in the literature where nonparametric rates differ for the pointwise mean-squared error and the mean integrated squared error. It is still an open question whether these rates are minimax optimal over all privacy mechanisms. Table 1 summarizes the rates attained using the non-interactive mechanism of Kroll (2024) and the rates attained using our sequentially interactive privacy mechanisms.

$\alpha-$LDP mechanism	$\mathbb{E}\|\hat{\sigma}_j - \sigma_j\|^2$	$\mathbb{E}\|\hat{f}(\omega) - f(\omega)\|^2$	$\mathbb{E}\\|\hat{f} - f\\|_2^2$
Non-interactive	$(n\alpha^4)^{-1}$	$(n\alpha^4)^{-\frac{2s}{2s+1}}$ if $f \in W^{s,\infty}(L_0, L)$ $(n\alpha^4)^{-\frac{2s-1}{2s}}$ if $f \in W^{s,2}(L)$	$(n\alpha^4)^{-\frac{2s}{2s+1}}$
Sequentially interactive	$(n\alpha^2)^{-1}$	$(n\alpha^2)^{-\frac{2s}{2s+1}}$ if $f \in W^{s,\infty}(L_0, L)$ $(n\alpha^2)^{-\frac{2s-1}{2s}}$ if $f \in W^{s,2}(L)$	$(n\alpha^2)^{-\frac{2s}{2s+2}}$ $\wedge (n\alpha^4)^{-\frac{2s}{2s+1}}$

Table 1: Convergence rates up to log-factors for recovering a single covariance coefficient σ_j, the spectral density f at a fixed point ω and globally. Here, $a \wedge b := \min\{a, b\}$.

Using the close connection between the spectral density f and the covariance matrix $\Sigma = (\sigma_{|i-j|})_{i,j=1,\ldots,n}$ for stationary processes, we also provide an estimator $\check{\Sigma}$ and convergence rates with respect to the Hilbert-Schmidt norm under α-LDP constraints.

These results confirm the importance of using sequentially interactive privacy mechanisms for achieving local differential privacy. Faster convergence rates with interactive mechanisms in time-series settings are intuitive, as past observations provide more valuable information than in i.i.d. settings. A key reason for the improvement in our setting is that the quantities of interest are second-order functionals. Non-interactive mechanisms privatize each observation X_i separately. As a result, products of privatized data lead to error terms involving products of independent privacy noises, whose variance scales with the fourth moment of the noise and yields an α^4 rate. In contrast, interactive mechanisms allow privatizing statistics closer to the target second-order functional, for instance the product of X_i and previously privatized data. In this case, the dominant error depends only on the second moment of the privacy noise, resulting in an α^2 rate.

Finally, we show that for estimating a single covariance coefficient σ_j, the rate $(n\alpha^4)^{-1}$ is minimax optimal over all non-interactive privacy mechanisms and estimators. In order to do that, we prove a general information-theoretic bound for the transcripts $Z_{1:n}$ of the sensitive data through an arbitrary privacy mechanism. The starting point is that the Fisher information of the parameter (here it is σ_j) contained in a sample $Z_{1:n} = (Z_1, ..., Z_n)$ also satisfies a chain rule:

$$I(Z_{1:n}) = I(Z_1) + I(Z_2|Z_1) + \ldots + I(Z_n|Z_{1:(n-1)}),$$

where, for all i from 1 to n, $I(Z_i|Z_{1:(i-1)})$ denotes the Fisher information of the conditional distribution, averaged over the conditioning variables. Lemma 9 shows that for all $i = 1, \ldots, n$ it holds that

$$I(Z_i|Z_{1:(i-1)}) \lesssim (e^{\alpha} - 1)^2 I(X_i|Z_{1:(i-1)}),$$

where, by convention, there is no conditioning when $i = 1$. Moreover, we have that

$$I(X_i|Z_{1:(i-1)}) \lesssim (e^{\alpha} - 1)^2 I(X_i, X_{i-1}|Z_{1:(i-2)}) + I(X_i|Z_{1:(i-2)}).$$

This lemma holds for any privacy mechanism, whether interactive or not. It is, however, applied to non-interactive mechanisms in conjunction with a particular choice of distribution for the sensitive variables, in order to show the minimax lower bound of order $1/(n\alpha^4)$ for privately estimating a single covariance coefficient σ_j. This clarifies that non-interactive mechanisms cannot attain faster minimax rates. As we further explain in Section 3.2, the above inequality shows that only a small amount of information propagates through a non-interactive mechanism, and this step does not extend to interactive mechanisms.

2. Building Private Samples and Estimators

Let $(X_1, ..., X_n)$ be a sample from a real-valued, centered stationary Gaussian time series with sequence of covariances $(\sigma_j)_{j\in\mathbb{Z}}$ and spectral density function $f : [-\pi, \pi] \to \mathbb{R}_{\geq 0}$ as defined in (1). We assume that the spectral density function is smooth, that is, f either belongs to the periodic Sobolev class of functions

$$W^{s,2}(L) := \left\{ f(\cdot) = \sum_{j\in\mathbb{Z}} \sigma_j \exp(-\mathtt{i}j\cdot) : \sum_{j\in\mathbb{Z}} |\sigma_j|^2 |j|^{2s} \leq L \right\},$$

where $s > 1/2$, or to the periodic Hölder class of functions

$$W^{s,\infty}(L_0, L) := \left\{ f(\cdot) = \sum_{j\in\mathbb{Z}} \sigma_j \exp(-\mathtt{i}j\cdot) : \|f\|_\infty < L,\ f^{(0)}, ..., f^{(\ell-1)} \text{ continuous and} \right.$$

$$|f^{(\ell)}(x) - f^{(\ell)}(y)| \leq L_0 |x-y|^{s-\ell} \, \forall \, x, y \in [-\pi, \pi] \Bigg\},$$

where $s > 1/2$, $\ell = \lfloor s \rfloor$ is the largest integer strictly less than s and $L_0, L > 0$ are some constants. The corresponding norms are denoted as $\| \cdot \|_2$ for the L_2 norm and $\| \cdot \|_\infty$ for the L_∞ norm. Furthermore, we use the notation $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$ for $a, b \in \mathbb{R}$. To simplify, we sometimes omit constant factors and use $\lesssim$ for less than or equal up to constants. The notation $a \asymp b$ is used to indicate that a and b grow at the same asymptotic rate, i.e., there are constants $c_1, c_2 > 0$ such that $c_1 b \leq a \leq c_2 b$.

Note that, for $s > 1/2$, $f \in W^{s,2}(L)$ implies that

$$\sum_{j \in \mathbb{Z}} \sigma_j^2 \leq L$$

and for $f \in W^{s,2}(L)$ or $f \in W^{s,\infty}(L_0, L)$, there exists a constant $M > 0$ such that

$$\sum_{j \in \mathbb{Z}} |\sigma_j| < M, \tag{2}$$

see Katznelson (2004).

Next, we introduce the definition of $\alpha-$local differential privacy (see, e.g., Duchi et al., 2018). In this framework, private samples $Z_1, ..., Z_n$ are successively obtained by drawing from conditional distributions $Q_i(\cdot \mid X_i, Z_1, \ldots, Z_{i-1})$, $i = 1, ..., n$. Let $\alpha \in (0, \infty)$. We say that a sequence of conditional distributions $(Q_i)_{i=1,...,n}$ provides $\alpha-$local differential privacy ($\alpha-$LDP) or that $Z_1, ..., Z_n$ are $\alpha-$local differentially private views of $X_1, ..., X_n$ if

$$\sup_{Z_1, \ldots, Z_{i-1}, X_i, X_i'} \frac{Q_i(\cdot \mid X_i, Z_1, \ldots, Z_{i-1})}{Q_i(\cdot \mid X_i', Z_1, \ldots, Z_{i-1})} \leq e^\alpha, \quad \text{for all } i = 1, \ldots, n, \tag{3}$$

where we use the convention that $\{Z_1, \ldots, Z_{i-1}\}$ is the empty set for $i = 1$. In particular, the ratio in Equation (3) quantifies the sensitivity to changes in the input. By bounding this ratio, one ensures that the probability of observing the same private output does not differ significantly given the previously privatized data and the two different inputs X_i and X_i'. The parameter α quantifies the privacy of the individual: the smaller the value of α, the stronger the privacy guarantee. Note that Z_i may also be a vector. As each Q_i is allowed to use previously released samples $Z_1, \ldots, Z_{i-1}$, we call this privacy mechanism *sequentially interactive*.

A subclass of LDP mechanisms is *non-interactive*, which are not allowed to use previously released data. A non-interactive $\alpha-$LDP privacy mechanism generates each Z_i solely based on X_i, via a conditional distribution of the form $Q_i(\cdot \mid X_i)$ satisfying Equation 3.

Local differential privacy is preserved under deterministic post-processing. Therefore, deterministic functions of the privatized samples, such as the estimators considered in this paper, satisfy the same privacy guarantee. Furthermore, estimators of nonnegative quantities may be replaced by their positive-part versions without affecting the privacy guarantees. For the estimation problems considered in this paper, this modification also does not affect the established risk bounds.

2.1 Non-interactive Privacy Mechanism

In this section, we give new results on the convergence rate of $\hat{\sigma}_j^{\mathrm{NI}}$ and $\hat{f}^{\mathrm{NI}}(\omega)$, which are estimators of $f(\omega)$ and σ_j obtained with the non-interactive $\alpha-$LDP mechanism of Kroll (2024). Let $(X_1, ..., X_n)$ be a sample from a Gaussian stationary time series $(X_t)_{t\in\mathbb{Z}}$ with mean zero and a spectral density such that $f \in W^{s,\infty}(L_0, L)$ or $f \in W^{s,2}(L)$. Let $\tau_n > 0$ be a threshold depending on the sample size n. The original data X_i is first truncated by setting

$$\widetilde{X_i} = (X_i \wedge \tau_n) \vee (-\tau_n), \quad i = 1, ..., n.$$

Subsequently, it is perturbed by adding a Laplace distributed random variable ξ_i with $\xi_1, ..., \xi_n \overset{\text{i.i.d.}}{\sim} \mathrm{Lap}(2\tau_n/\alpha)$, yielding

$$Z_i = \widetilde{X_i} + \xi_i, \quad i = 1, ..., n.$$

This mechanism is non-interactive and satisfies the definition of $\alpha-$LDP in Equation (3). Define the bias corrected periodogram by

$$\hat{I}_n(\omega) = \frac{1}{2\pi n}\left|\sum_{t=1}^{n} Z_t \exp(-\mathtt{i}t\omega)\right|^2 - \frac{4\tau_n^2}{\pi\alpha^2}.$$

Then, a spectral density estimator is defined by the partial Fourier sum of order $m \in \mathbb{N}$

$$\hat{f}_m^{\mathrm{NI}}(\cdot) = \frac{1}{2\pi}\sum_{j=-m}^{m} \hat{\sigma}_j^{\mathrm{NI}} \exp(-\mathtt{i}j\cdot), \text{ where } \hat{\sigma}_j^{\mathrm{NI}} = \int_{-\pi}^{\pi} \hat{I}_n(\omega)\exp(\mathtt{i}j\omega)\,\mathrm{d}\omega.$$

Kroll (2024) showed that if $m \asymp \{1/n \vee \tau_n^4/(n\alpha^4)\}^{-\frac{1}{2s+1}}$ and $\tau_n^2 = 56\log^{1+\delta}(n)$ for some constant $\delta > 0$, then, for some constant $c > 0$ and sufficiently large n

$$\mathbb{E}\|\hat{f}_m^{\mathrm{NI}} - f\|_2^2 \leq c\left[\left\{\frac{1}{n} \vee \frac{\log^{2+2\delta}(n)}{n\alpha^4}\right\}^{\frac{2s}{2s+1}} \vee \frac{\log^{2+2\delta}(n)}{n}\right].$$

We complement this result by proving the convergence rates for the estimation of the spectral density $\hat{f}_m^{\mathrm{NI}}(\omega)$ at a fixed frequency ω and of a single covariance coefficient $\hat{\sigma}_j^{\mathrm{NI}}$.

Proposition 1 *Let $(X_t)_{t\in\mathbb{Z}}$ be a centered stationary Gaussian time series such that $\|\sigma\|_2^2 = \sum_{k\in\mathbb{Z}} \sigma_k^2 \leq M_1$ for some constant $M_1 > 0$. Define $\tau_n^2 = 56\log^{1+\delta}(n)$ for some $\delta > 0$. Then, for sufficiently large n, the following holds*

i) Let $R \in \mathbb{N}$ such that $R \leq \sqrt{n}$. For every $j \in \{-R, ..., R\}$, it holds

$$\sup_{\sigma:\,\|\sigma\|_2^2 \leq M_1} \mathbb{E}|\hat{\sigma}_j^{NI} - \sigma_j|^2 \leq c_1 \log^{2+2\delta}(n)\left\{\frac{1}{n} \vee \frac{1}{n\alpha^4}\right\},$$

where the constant $c_1 > 0$ depends on M_1.

ii) If $m \asymp (\frac{1}{n} \vee \frac{\tau_n^4}{n\alpha^4})^{-\frac{1}{2s+1}}$*, then for every* $\omega \in [-\pi, \pi]$*, it holds that*

$$\sup_{f \in W^{s,\infty}(L_0, L)} \mathbb{E}|\hat{f}_m^{NI}(\omega) - f(\omega)|^2 \le c_2 \left[\left\{ \frac{\log(n)}{n} \vee \frac{\log^{3+2\delta}(n)}{n\alpha^4} \right\}^{\frac{2s}{2s+1}} \vee \frac{\log^{2+2\delta}(n)}{n} \right],$$

where the constant $c_2 > 0$ *depends on* s*,* L_0, L *and* M*, with* M *given in (2).*

iii) If $m \asymp (\frac{1}{n} \vee \frac{\tau_n^4}{n\alpha^4})^{-\frac{1}{2s}}$*, then for every* $\omega \in [-\pi, \pi]$*, it holds that*

$$\sup_{f \in W^{s,2}(L)} \mathbb{E}|\hat{f}_m^{NI}(\omega) - f(\omega)|^2 \le c_3 \left[\left\{ \frac{1}{n} \vee \frac{\log^{2+2\delta}(n)}{n\alpha^4} \right\}^{\frac{2s-1}{2s}} \vee \frac{\log^{2+2\delta}(n)}{n} \right],$$

where the constant $c_3 > 0$ *depends on* s*,* L *and* M*, with* M *given in (2).*

In particular, to estimate σ_j, only square-summability of the Fourier coefficients, but no further smoothness assumption on f is required. The result for $\hat{\sigma}_j^{\mathrm{NI}}$ holds for $j \in \{-(n-1), ..., (n-1)\}$ fixed or slowly increasing with n as long as $j = \mathcal{O}(n^{1/2})$. If the normalization factor is replaced by $n/(n-j)$, that is, if $\hat{\sigma}_j^{\mathrm{NI}} n/(n-j)$ is considered, then the result holds for $j = 0, 1, .., R$ with $0 < R < n$ and $R = o(n)$. Note that the exponent $(2s-1)/(2s)$ in the pointwise convergence rate for a Sobolev smooth spectral density function is optimal in the classical nonparametric (non-private) setting.

2.2 Interactive Privacy Mechanism for σ_j

In this section, a new interactive privacy mechanism and an estimator for the covariance coefficient σ_j are introduced. Since $\sigma_j = \sigma_{-j}$, it is sufficient to consider $j \in \{0, 1, ..., n-1\}$. Let $\xi_1, ..., \xi_n \overset{\text{i.i.d.}}{\sim} \mathrm{Lap}(4\tau_n/\alpha)$ be a set of Laplace random variables. If $j \neq 0$, then for the first j iterations, $i = 1, ..., j$, the i-th data holder truncates its observation X_i at a threshold $\tau_n > 0$ and adds Laplace noise ξ_i, yielding

$$Z_i = \widetilde{X}_i + \xi_i, \quad \text{where } \widetilde{X}_i = (X_i \wedge \tau_n) \vee (-\tau_n). \tag{4}$$

Consider a second set of Laplace random variables: $\tilde{\xi}_{j+1}, ..., \tilde{\xi}_n \overset{\text{i.i.d.}}{\sim} \mathrm{Lap}(4\tilde{\tau}_n/\alpha)$ for some $\tilde{\tau}_n > 0$, which is independent of $\xi_1, ..., \xi_n$ and $X_1, ..., X_n$. Starting from iteration $i \geq j+1$, we build both Z_i as before and a second random variable $\bar{Z}_{i,j}$ by

$$\bar{Z}_{i,j} = \widetilde{W}_{i,j} + \tilde{\xi}_i, \quad \text{where } \widetilde{W}_{i,j} = (W_{i,j} \wedge \tilde{\tau}_n) \vee (-\tilde{\tau}_n) \text{ and } W_{i,j} = X_i \cdot Z_{i-j}. \tag{5}$$

In the case of $j = 0$, we build directly, for all $i = 1, \ldots, n$, the random variables

$$\bar{Z}_{i,0} = \widetilde{W}_{i,0} + \xi_i, \quad \text{where } \widetilde{W}_{i,0} = (X_i^2 \wedge \tau_n) \vee (-\tau_n)$$

and $\xi_1, ..., \xi_n \overset{\text{i.i.d.}}{\sim} \mathrm{Lap}(2\tau_n/\alpha)$. The proposed mechanism is sequentially interactive (for $j \neq 0$) and satisfies the definition of $\alpha-$local differential privacy as shown in the following lemma.

Lemma 2 *Let $j \in \{0, 1, ..., n\}$. Then, for $i = 1, ..., j$, Z_i is an $\alpha-$local differentially private view of X_i, and for $i = j+1, ..., n$, $(Z_i, \bar{Z}_{i,j})$ is an $\alpha-$local differentially private view of X_i.*

The covariance coefficient σ_j is then estimated by

$$\hat{\sigma}_j = \frac{1}{n-j} \sum_{i=j+1}^{n} \bar{Z}_{i,j}. \tag{6}$$

We show that the mean-squared error of $\hat{\sigma}_j$ is of the order $1/(n\alpha^2)$. This is an improvement compared to the convergence rate in Proposition 1 for the non-interactive privacy mechanism, which depends on α^4. As in Proposition 1, to estimate σ_j, no smoothness assumption on f is required.

Theorem 3 *Let $(X_t)_{t\in\mathbb{Z}}$ be a centered stationary Gaussian time series such that $\|\sigma\|_2^2 = \sum_{k\in\mathbb{Z}} \sigma_k^2 \leq M_1$ for some constant $M_1 > 0$. Define $\tau_n^2 = 8\log^{1+\delta}(n)$ and $\tilde{\tau}_n = 16\log^{1+\delta}(n)\tau_n^2$ for some $\delta > 0$. Let $R \in \mathbb{N}$ such that $0 < R < n$ and $R = o(n)$. Then, for $j \in \{0, 1, \ldots, R\}$ and some constant $c > 0$, the estimator $\hat{\sigma}_j$ of σ_j, as defined in (6), satisfies for sufficiently large n that*

$$\sup_{\sigma:\, \|\sigma\|_2^2 \leq M_1} \mathbb{E}|\hat{\sigma}_j - \sigma_j|^2 \leq c \left\{ \frac{1}{n} \vee \frac{\log^{4+4\delta}(n)}{n\alpha^2} \right\},$$

where the constant $c > 0$ depends on M_1.

In particular, the result for $\hat{\sigma}_j$ holds for any $j \in \{-R, ..., R\}$ fixed as long as $R = o(n)$.

2.3 Interactive Privacy Mechanism for Pointwise Estimation

In this section, an interactive privacy mechanism and an estimator for the spectral density function at a fixed $\omega \in [-\pi, \pi]$ are introduced. Let $K \in \mathbb{N}$ be such that $K \ll n$. For the first K iterations, we proceed as in Section 2.2, i.e., truncating X_i with some threshold $\tau_n > 0$ and adding an independent Laplace noise $\xi_i \sim \text{Lap}(4\tau_n/\alpha)$ to obtain Z_i in (4). Starting from iteration $i \geq K+1$, we build both Z_i in (4) and a second random variable $\widetilde{Z}_i$ as follows. Let

$$V_i = X_i^2 + \sum_{1\leq|k|\leq K} a_k X_i Z_{i-|k|} \exp(-\mathtt{i}\omega k), \tag{7}$$

where $a_k = 1$ for $0 \leq |k| \leq K/2$ and $a_k = 2(1 - |k|/K)$ for $K/2 < |k| \leq K$. Next, V_i is trimmed at the level $\tilde{\tau}_n$, for some threshold $\tilde{\tau}_n > 0$, and we build $\widetilde{Z}_i$ by adding a Laplace distributed random variable $\tilde{\xi}_i$ with $\tilde{\xi}_{K+1}, ..., \tilde{\xi}_n \overset{\text{i.i.d.}}{\sim} \text{Lap}(4\tilde{\tau}_n/\alpha)$, yielding

$$\widetilde{Z}_i = \widetilde{V}_i + \tilde{\xi}_i, \quad \text{where } \widetilde{V}_i = (V_i \wedge \tilde{\tau}_n) \vee (-\tilde{\tau}_n). \tag{8}$$

The proposed mechanism is sequentially interactive and satisfies the definition of $\alpha-$local differential privacy as shown in the following lemma.

Lemma 4 *For $i = 1, ..., K$, Z_i is an $\alpha-$local differentially private view of X_i. For $i = K+1, ..., n$, $(Z_i, \widetilde{Z}_i)$ is an $\alpha-$local differentially private view of X_i.*

The estimator of $f(w)$ is then defined by

$$\hat{f}_K(\omega) = \frac{1}{2\pi(n-K)} \sum_{i=K+1}^{n} \widetilde{Z}_i.$$

Note that the random variables $V_i = V_i(\omega)$ and $\widetilde{Z}_i = \widetilde{Z}_i(\omega)$ depend on the point of interest ω. The next theorem shows that the estimator $\hat{f}_K(\omega)$ based on the sequentially interactive privatization mechanism attains the usual nonparametric convergence rates in the non-private setting up to log-factors and a loss of $1/\alpha^2$. Compared to Proposition 1, the loss of $1/\alpha^4$ for non-interactive privacy mechanisms is improved to the smaller loss of $1/\alpha^2$ as α tends to 0.

Theorem 5 *Let $\omega \in [-\pi, \pi]$. Define $\tau_n^2 = 8\log^{1+\delta}(n)$ and $\tilde{\tau}_n^2 = 1024\tau_n^6(K+1)$ for some $\delta > 0$. Then, it holds for sufficiently large n that*

i) If $K \asymp \{1/n \vee \tau_n^6/(n\alpha^2)\}^{-\frac{1}{2s+1}}$, then

$$\sup_{f \in W^{s,\infty}(L_0,L)} \mathbb{E}|\hat{f}_K(\omega) - f(\omega)|^2 \leq c_1 \left\{ \frac{1}{n} \vee \frac{\log^{3+3\delta}(n)}{n\alpha^2} \right\}^{\frac{2s}{2s+1}},$$

where the constant $c_1 > 0$ depends on s, L_0, L and M, with M given in (2).

ii) If $K \asymp \{1/n \vee \tau_n^6/(n\alpha^2)\}^{-\frac{1}{2s}}$, then

$$\sup_{f \in W^{s,2}(L)} \mathbb{E}|\hat{f}_K(\omega) - f(\omega)|^2 \leq c_2 \left\{ \frac{1}{n} \vee \frac{\log^{3+3\delta}(n)}{n\alpha^2} \right\}^{\frac{2s-1}{2s}},$$

where the constant $c_2 > 0$ depends on s, L and M, with M given in (2).

Note that the weights a_k in the definition of V_i in Equation (7) result in a reduced bias for Hölder-smooth functions. Consequently, the first factor in the rate in Theorem 5 *i)* is n^{-1} rather than $\log(n)n^{-1}$, as stated in Proposition 1 *ii)*.

2.4 Interactive Privacy Mechanism for Global Estimation

There are two possible extensions of the methods in Section 2.2 and 2.3 to design an $\alpha-$LDP mechanism and an estimator for the whole spectral density f: either by building a $(K+1)$-dimensional vector of $W_{i,j}$, $j = 0, ..., K$, to estimate K covariance coefficients $\sigma_0, ..., \sigma_K$ in the i-th iteration. Or, a vector of $V_i(\omega_k), k = 1, ..., N$, is built in the i-th iteration to estimate the spectral density on an N-dimensional grid $(\omega_k)_{k=1}^N$ of $[0, \pi]$ and then to smooth it. The choice of privatization of the vector is crucial. Using coordinate-wise Laplace noise, the variance scales with K, and a suboptimal exponent of $2s/(2s+3)$ in the first case and of $2s/(2s+4)$ in the second case can be derived. Therefore, we employ the mechanism of Duchi et al. (2018) to privatize vectors in an ℓ_∞ ball. For the first K iterations, we proceed similarly as in Section 2.2, i.e., we truncate at threshold $\tau_n > 0$ and add independent Laplace noise $\xi_i \sim \text{Lap}(4\tau_n/\alpha)$ such that

$$Z_i = \widetilde{X}_i + \xi_i, \quad \text{where } \widetilde{X}_i = (X_i \wedge \tau_n) \vee (-\tau_n).$$

Starting from iteration $i \geq K+1$, we build both Z_i as above and the vector

$$\widetilde{W}_i = (\widetilde{W}_{i,0}, \ldots, \widetilde{W}_{i,K})^\top \in [-\tilde{\tau}_n, \tilde{\tau}_n]^{K+1},$$

where

$$\widetilde{W}_{i,j} = (W_{i,j} \wedge \tilde{\tau}_n) \vee (-\tilde{\tau}_n) \text{ and } \begin{cases} W_{i,j} = X_i \cdot Z_{i-j}, & j = 1, ..., K, \\ W_{i,0} = X_i^2, & j = 0, \end{cases}$$

with some threshold $\tilde{\tau}_n > 0$. In particular, $\widetilde{W}_i \in B_\infty(\tilde{\tau}_n)$, where $B_\infty(r) = \{x \in \mathbb{R}^{K+1} \mid \|x\|_\infty \leq r\}$ is the ℓ_∞-ball of radius r. We apply the privacy mechanism for ℓ_∞ balls of Duchi et al. (2018), corrected by Butucea et al. (2023). This mechanism applies to multivariate vectors whose sup-norm is bounded by r. The outcomes are the vertices of a hypercube with coordinates in $\{\pm B\}^{K+1}$ which are split in two sets according to the value of α. Duchi et al. (2018) showed that a Laplace mechanism applied to each entry of the vector will introduce too much variance in the transcripts, a factor K that makes all estimators based on these randomizations suboptimal. The hypercube mechanism we briefly describe here reduces the variance to its minimum and yields optimal estimators.

To obtain a private view $\check{Z}_i \in \mathbb{R}^{K+1}$ of $\widetilde{W}_i$ in the i-th iteration, we perform the following steps:

- If $\widetilde{W}_i = (w_0, ..., w_K)^\top$, define a random vector $\widetilde{Y}_i = (\widetilde{Y}_{i,0}, ..., \widetilde{Y}_{i,K})^\top$ with

$$\widetilde{Y}_{i,j} := \begin{cases} +\tilde{\tau}_n, & \text{with probability } \frac{1}{2} + \frac{w_j}{2\tilde{\tau}_n}, \\ -\tilde{\tau}_n, & \text{otherwise.} \end{cases}$$

- Sample $T_i \sim Ber(\pi_\alpha)$, where $\pi_\alpha = e^\alpha/(e^\alpha + 1)$, and set

$$\widetilde{Z}_i := \begin{cases} \text{Uniform}\left\{ z \in \{\pm B\}^{K+1} \mid \langle z, \widetilde{Y}_i \rangle > 0 \text{ or } \left(\langle z, \widetilde{Y}_i \rangle = 0 \text{ and } \tilde{z}_1 = \frac{B}{\tilde{\tau}_n} \widetilde{Y}_{i,0} \right) \right\}, & \text{if } T_i = 1, \\ \text{Uniform}\left\{ z \in \{\pm B\}^{K+1} \mid \langle z, \widetilde{Y}_i \rangle < 0 \text{ or } \left(\langle z, \widetilde{Y}_i \rangle = 0 \text{ and } z_1 = \frac{-B}{\tilde{\tau}_n} \widetilde{Y}_{i,0} \right) \right\}, & \text{if } T_i = 0, \end{cases}$$

  where B is chosen as

$$B = \tilde{\tau}_n \frac{e^\alpha + 1}{e^\alpha - 1} C_K, \text{ where } \frac{1}{C_K} = \begin{cases} \frac{1}{2^K} \binom{K}{\frac{K}{2}}, & \text{if } K+1 \text{ is odd}, \\ \frac{(K-1)!(K-1)}{2^K (\frac{K+1}{2} - 1)! \frac{K+1}{2}!}, & \text{if } K+1 \text{ is even}. \end{cases}$$

- Define the vector $\check{Z}_i$ by $\check{Z}_i = \widetilde{Z}_i$ if $K+1$ is odd, and by its components

$$\check{Z}_{i,j} = \begin{cases} \frac{K-1}{2K} \widetilde{Z}_{i,0}, & \text{if } j = 0, \\ \widetilde{Z}_{i,j}, & \text{if } j = 1, \ldots, K, \end{cases}$$

  if $K+1$ is even.

We show that this mechanism satisfies the definition of $\alpha-$LDP.

Lemma 6 *The described mechanism has the following properties.*

i) For all $i = 1, ..., K$, Z_i is an $\alpha-$local differentially private view of X_i, and for all $i = K+1, ..., n$, the sample $(Z_i, \check{Z}_i)$ is an $\alpha-$local differentially private view (X_i, W_i).

ii) For all $i = K+1, \dots, n$ and $j = 0, ..., K$, it holds

$$\begin{aligned} & \mathbb{E}[\check{Z}_i \mid X_i, Z_{1:(i-1)}, \check{Z}_{1:(i-1)}] = \mathbb{E}[\check{Z}_i \mid X_i, Z_{1:(i-1)}] = \widetilde{W}_i \\ \text{and} \quad & \operatorname{Var}(\check{Z}_{i,j} \mid X_i, Z_{1:(i-1)}, \check{Z}_{1:(i-1)}) = \operatorname{Var}(\check{Z}_{i,j} \mid X_i, Z_{1:(i-1)}) \le B^2, \end{aligned}$$

where $B \lesssim \tilde{\tau}_n \sqrt{K+1}/\alpha$, if $\alpha \in (0,1)$.

The estimator of f is then defined by

$$\check{f}_K(\omega) = \frac{1}{2\pi(n-K)} \sum_{i=K+1}^{n} \sum_{0 \le |k| \le K} \check{Z}_{i,|k|} \exp(-\mathtt{i} k\omega), \quad \omega \in [-\pi, \pi],$$

where $\check{Z}_{i,k}$ is the k-th entry of $\check{Z}_i$, for $k = 0, ..., K$.

Finally, we show that the convergence rate of this estimator depends on α^2, but with a loss in the exponent.

Theorem 7 *Define $\tau_n^2 = 8\log^{1+\delta}(n)$ and $\tilde{\tau}_n = 16\log^{1+\delta}(n)\tau_n^2$ for some $\delta > 0$. If $K \asymp \{\tilde{\tau}_n^2/(n\alpha^2)\}^{-1/(2s+2)} \wedge n^{-1/(2s+1)}$, then for sufficiently large n*

$$\begin{aligned} & \sup_{f \in W^{s,\infty}(L_0,L)} \mathbb{E}\|\check{f}_K - f\|_2^2 \le c_1 \left(\frac{\log^{4+4\delta}(n)}{n\alpha^2}\right)^{\frac{2s}{2s+2}} \vee \left(\frac{1}{n}\right)^{\frac{2s}{2s+1}} \\ \text{and} \quad & \sup_{f \in W^{s,2}(L)} \mathbb{E}\|\check{f}_K - f\|_2^2 \le c_2 \left(\frac{\log^{4+4\delta}(n)}{n\alpha^2}\right)^{\frac{2s}{2s+2}} \vee \left(\frac{1}{n}\right)^{\frac{2s}{2s+1}}, \end{aligned}$$

where the constant $c_1 > 0$ depends on s, L, and the constant $c_2 > 0$ additionally on L_0.

Combining this with the result of Kroll (2024) gives the following upper bound on the L_2 risk over the class of Sobolev smooth spectral density function $W^{s,2}(L)$ for the low-privacy regime $\alpha \in (0,1)$:

$$\left(\frac{\log^{4+4\delta}(n)}{n\alpha^2}\right)^{\frac{2s}{2s+2}} \wedge \left(\frac{\log^{2+2\delta}(n)}{n\alpha^4}\right)^{\frac{2s}{2s+1}},$$

where the minimum depends on the magnitude of α. For small enough values of $\alpha = \alpha_n$, such that $n\alpha_n^{2(2s+3)} \log^{4s+4s\delta}(n)$ tends to 0, then our interactive procedure achieves faster rates than the non-interactive procedure.

The pointwise risk of the estimator $\check{f}_K$ is slower than the one of $\hat{f}_K(\omega)$ as shown in Theorem 5.

Corollary 8 *Define τ_n and $\tilde{\tau}_n$ as in Theorem 7. Then it holds for sufficiently large n that*

i) If $K \asymp \{\tilde{\tau}_n^2/(n\alpha^2)\}^{-1/(2s+2)} \wedge n^{-1/(2s+1)}$, then

$$\sup_{f \in W^{s,\infty}(L_0,L)} \mathbb{E}|\check{f}_K(\omega) - f(\omega)|^2 \le c_1 \left(\frac{\log^{5+4\delta}(n)}{n\alpha^2}\right)^{\frac{2s}{2s+2}} \vee \left(\frac{\log(n)}{n}\right)^{\frac{2s}{2s+1}},$$

where the constant $c_1 > 0$ depends on s, L_0, L and M, with M given in (2).

ii) If $K \asymp \{\tilde{\tau}_n^2/(n\alpha^2)\}^{-1/(2s+1)} \wedge n^{-1/(2s)}$, *then*

$$\sup_{f \in W^{s,2}(L)} \mathbb{E}|\check{f}_K(\omega) - f(\omega)|^2 \leq c_2 \left(\frac{\log^{4+4\delta}(n)}{n\alpha^2} \right)^{\frac{2s-1}{2s+1}} \vee \left(\frac{\log(n)}{n} \right)^{\frac{2s-1}{2s}},$$

where the constant $c_2 > 0$ *depends on* s, L *and* M*, with* M *given in (2).*

Let us remark that an estimator for the Toeplitz covariance matrix $\Sigma = (\sigma_{|i-j|})_{i,j=1,\dots,n}$ can be easily derived from our procedure. The benefit of building an estimator of the spectral density function is that it becomes easier to produce a bona fide covariance estimator. For this, one must truncate the negative values of the estimator: $\check{f}_K^+ := \check{f}_K \vee 0$, and then apply the Fourier transform to obtain the estimators $\sigma_j^\dagger$, $j = 0, \dots, n-1$. Since $\check{f}_K^+$ is non-negative and symmetric, the resulting plug-in estimator $\check{\Sigma} = (\sigma_{|i-j|}^\dagger)_{i,j=1,\dots,n}$ is a proper positive semi-definite Toeplitz matrix. By the relation of the Hilbert-Schmidt norm, denoted by $\|\cdot\|_{HS}$, and the L_2 norm, it follows

$$\mathbb{E}\|\check{\Sigma} - \Sigma\|_{HS}^2 \leq c \left(\frac{\log^{4+4\delta}(n)}{n\alpha^2} \right)^{\frac{2s}{2s+2}}.$$

3. Non-interactive Mechanisms are Necessarily Slower

The information theoretic inequalities by Duchi et al. (2018) were developed for independent initial random variables $X_1, \dots, X_n$. In our case, the dependence is at the core of the problem. Therefore, we develop in the next section a new information-theoretic inequality to hold for arbitrary multivariate likelihoods of the joint vector $X_{1:n} = (X_1, \dots, X_n)$ and any sequentially interactive (or non-interactive) mechanisms. The second section is an application of our inequality to show that our faster rates $1/(n\alpha^2)$ for estimating the covariance value σ_j cannot be attained by any non-interactive mechanisms since the best they can do is $1/(n\alpha^4)$. We use the new inequality of the first section in conjunction with a particular construction of a Gaussian time series $X_1, \dots, X_n$ in order to prove that only slower rates can be attained using non-interactive mechanisms.

3.1 Information Theoretic Inequality for Dependent Data

The Fisher information of the parameter θ contained in a sample $Z_{1:n} = (Z_1, \dots, Z_n)$ also satisfies a chain rule

$$I(Z_{1:n}) = I(Z_1) + I(Z_2|Z_1) + \ldots + I(Z_n|Z_{1:(n-1)}),$$

with, for all i from 1 to n,

$$I(Z_i|Z_{1:(i-1)}) := \mathbb{E}_{Z_{1:(i-1)}} \left[\int \frac{(\dot{\mathcal{L}}_\theta^{Z_i|Z_{1:(i-1)}}(z_i))^2}{\mathcal{L}_\theta^{Z_i|Z_{1:(i-1)}}(z_i)} \mathrm{d}z_i \right].$$

Lemma 9 *For all* $i = 1, \dots, n$*, we have*

$$I(Z_i|Z_{1:(i-1)}) \lesssim (e^\alpha - 1)^2 I(X_i|Z_{1:(i-1)}), \tag{9}$$

where, by convention, there is no conditioning when $i = 1$. Moreover, for an arbitrary $\epsilon > 0$,

$$I(X_i|Z_{1:(i-1)}) \leq (1 + \frac{1}{\epsilon})(e^{\alpha} - 1)^2 I(X_i, X_{i-1}|Z_{1:(i-2)}) + (1 + \epsilon)e^{2\alpha} I(X_i|Z_{1:(i-2)}). \tag{10}$$

and

$$I(X_i|Z_{1:(i-1)}) \leq (1 + \frac{9}{4}(e^{\alpha} - 1)^2) \left(I(X_i, X_{i-1}|Z_{1:(i-2)}) + I(X_i|Z_{1:(i-2)}) \right). \tag{11}$$

Note that the coefficients of the terms on the right-hand side in Equation (10) are much smaller for the first term and decrease to 1 for the second as $\alpha > 0$ tends to 0, whereas in (11) the coefficients are balanced and decrease to 1 as α tends to 0. We think both versions may be useful in different contexts.

3.2 Non-interactive Mechanisms are Slower for Estimating a Covariance Coefficient

By the results of Kroll (2024), $1/(n\alpha^2)$ is a lower bound for estimating σ_0. Here, we show that for estimating σ_j for some fixed $j \geq 1$ using non-interactive privacy mechanisms, the lower bound increases to $1/(n\alpha^4)$, showing optimality of the plug-in estimator of Kroll (2024) based on a non-interactive privacy mechanism. However, we recall that our interactive procedure and the associated estimator attain the faster estimation rate $1/(n\alpha^2)$.

Let us denote by $\mathcal{Q}_{\alpha}^{\mathrm{NI}}$ the class of non-interactive privacy mechanisms which are $\alpha-$LDP. We assume here that $0 < \alpha \leq \alpha_0 < 1$ for some fixed α_0.

Theorem 10 *Assume X is an $n-$dimensional vector generated from a centered stationary Gaussian time series with $\|\sigma\|_2^2 = \sum_{j\in\mathbb{Z}} \sigma_j^2 \leq M_1$. Then, for any arbitrary, fixed $K \geq 1$, we have*

$$\inf_{Q\in\mathcal{Q}_{\alpha}^{\mathrm{NI}}} \inf_{\hat{\sigma}_K} \sup_{\sigma:\|\sigma\|_2^2\leq M_1} \mathbb{E}\left[|\hat{\sigma}_K - \sigma_K|^2\right] \geq \frac{C}{n\alpha^4},$$

for sufficiently large n.

Thus, the faster rate $1/(n\alpha^2)$ that our interactive method attains cannot be obtained by any non-interactive method.

Sketch of proof of Theorem 10: We proceed by reducing the set of parameters to a simple $K-$dependent centered and stationary Gaussian vector, that is, only σ_0 and σ_K are non-zero. We let σ_K vary in a small interval and apply the Van Trees inequality to obtain the first bound from below. The bound involves the Fisher information that the privatized vector $Z_{1:n}$ brings on the parameter of interest. After the telescopic expansion of this Fisher information for conditional likelihoods, Lemma 9 provides a one-step contraction relating $I(Z_i|Z_{1:(i-1)})$ to $I(X_i|Z_{1:(i-1)})$ under local differential privacy. In order to obtain an α^{-4} lower bound for non-interactive mechanisms, we additionally control the latter information by using the fact that each private sample Z_i only depends on the sensitive X_i at each time step i and not on the previous transcripts $Z_1, ..., Z_{i-1}$. This restricts how much information about the dependence parameter can propagate through the transcript and yields an additional α^2-order contraction. This additional step does not apply to interactive mechanisms because Z_i may adapt to past releases, thereby making the transcript more informative about the correlation parameters.

4. Simulation Study

We compare the finite-sample performance of the proposed sequentially interactive (SI) private estimators with the non-interactive (NI) private estimators of Kroll (2024) for estimating a covariance coefficient and the spectral density at a fixed frequency. In particular, we focus on the dependence of the mean squared error (MSE) on the privacy level α for a fixed, sufficiently large sample size n, to validate the privacy–accuracy trade-off predicted by our theoretical results in the previous sections. We consider three mean-zero Gaussian stationary processes $(X_1, \ldots, X_n)$ with $n = 1000$ and variance 1.44:

(1) an autoregressive process AR(0.8) with Gaussian innovations and variance 1.44,

(2) a Gaussian process with spectral density $f(x) = 0.2\{|\cos(x)|^{0.8} + 0.45\}$ for $x \in \mathbb{R}$,

(3) a Gaussian process with covariance coefficients $\sigma_k = 1.44(1+|k|)^{-5.1}$ for $k = 0, ..., n-1$.

The three examples are chosen to represent Gaussian stationary processes with substantially different dependence structures. Example (1) has exponentially decaying covariance coefficients and an analytic spectral density, Example (2) has slowly decaying covariances $\sigma_k = \mathcal{O}(k^{-0.8})$ and a 0.8-Hölder-continuous spectral density, and Example (3) exhibits faster polynomially decaying covariances and a moderately smooth spectral density $f \in W^{4,\infty}$.

We estimate the variance σ_0, the lag-2 covariance coefficient σ_2, and the spectral density at frequency $\omega = \pi/5$. Performance is measured by the Monte Carlo estimate of the mean squared error, computed over 300 independent replications. For spectral density estimation, we use $K = \lceil 3\{n \vee (n\alpha^2)/\tau_n^6\}^{1/(2s+1)} \rceil$ in the sequentially interactive case and $m = \lceil 3\{n \vee (n\alpha^4)/\tau_n^4\}^{1/(2s+1)} \rceil$ in the non-interactive case, where $\lceil \cdot \rceil$ denotes the ceiling function. We set $s = 3$ and $\delta = 0.001$ for all examples, as the influence is negligible, provided δ is chosen small. Negative variance estimates are replaced by 10^{-3}, and negative spectral density estimates are truncated to zero. For comparison, we furthermore estimate the MSE of the corresponding standard estimators on the original data, which are $\hat{\sigma}_j = \frac{1}{n}\sum_{k=1}^{n-j} X_k X_{k+j}$ and $f(x) = \frac{1}{2\pi}\sum_{j=-K}^{K} \hat{\sigma}_j \exp(-\mathtt{i}jx)$ with $K = \lceil 3n^{1/(2s+1)} \rceil$.

We present the results for Example (1) in detail. The remaining examples are reported in the Appendix and lead to similar conclusions. Figure 1 displays the logarithm of the estimated MSE as a function of $\log(\alpha)$ for NI-based estimators (left) and SI-based estimators (right), for σ_0, σ_2 and $f(\pi/5)$. The logarithm of the theoretical MSE in the privacy-dominated regime is overlaid as dashed gray lines with slopes -4 for NI-based estimation and -2 for SI-based estimation.

Using the theoretical truncation levels τ_{theory} and $\tilde{\tau}_{\text{theory}}$ as defined in Proposition 1, Theorem 3, and Theorem 5, the logarithm of the empirical MSE closely follows the predicted slopes for small and moderate values of α. As α increases further, the MSE coincides with the MSE of the corresponding non-private estimator. Overall, the simulations confirm the theoretical scaling of the MSE with respect to α.

For both NI-based and SI-based estimators, the theoretical truncation levels tend to be too large in finite samples, leading to very large MSE for strong privacy (small α). In practice, moderate reductions of τ_{theory} and $\tilde{\tau}_{\text{theory}}$ can substantially reduce the MSE for strong privacy levels without affecting the α-rate. For NI-based variance estimation, reducing $\tau^{\text{NI}}_{\text{theory}} \approx 20$ to $\tau \approx 3$ uniformly improves or preserves performance across α. In contrast, more aggressive truncation, that is, choosing τ close to or smaller than the true parameter value, introduces a truncation bias and leads to an increase in the MSE for larger values of α. For estimation of σ_2 and $f(\pi/5)$ from NI-privatized data, similar qualitative behavior is observed. For SI-based variance estimation, the theoretical truncation level is close to the boundary at which truncation bias becomes visible. Slightly smaller truncation values reduce the MSE for $\alpha \in (0, 1)$ but lead to a bias for larger α. For estimation of σ_2 and $f(\pi/5)$ from SI-privatized data, a moderate simultaneous reduction of the truncation parameters τ and $\tilde{\tau}$ leads to a noticeable decrease of the MSE over a wide range of α, while overly strong truncation again introduces a bias for larger α. No substantial difference in sensitivity between τ and $\tilde{\tau}$ is observed in this setting.

Across all three estimation problems, the SI-based estimators achieve substantially lower MSE than their NI counterparts in the strong privacy regime, consistent with the improved scaling in α. For variance estimation, the SI-based estimator outperforms the NI-based estimator uniformly over α, whereas for covariance and pointwise spectral density estimation, the advantage is most pronounced for $\alpha \in (0, 1)$, with the precise crossover depending on the exact choice of the truncation parameters.

Increasing the sample size n yields similar observations regarding the alignment of estimated MSE with the theoretical results and the influence of the truncation parameter. If the variance of the underlying process increases, the true values of σ_0, σ_2 and $f(\pi/5)$ increase, and truncation bias becomes relevant earlier.

In Example (2) and (3), where σ_2 is close to zero, aggressive truncation reduces the variance of the estimator while introducing only negligible bias. As a result, the MSE is smaller than that of the non-private estimator for moderate and large α. Otherwise, the results for Examples (2) and (3) are qualitatively similar to those of Example (1), despite their different dependence structures. The simulations suggest that, in finite samples, truncation and privacy noise dominate the magnitude of the MSE under local differential privacy and not the dependence structure. The code for the simulation study is available on GitHub.

Appendix A. Proofs

Throughout this section, we use $c, c_1, C, C_1, \dots$ to denote generic constants that are independent of K, n, α. The indicator function is written as $\mathbf{1}(\text{condition})$, which equals 1 if the condition holds and 0 otherwise. By $\mathbb{I}_A$, we denote a random variable that takes the value 1 if the event A occurs and 0 otherwise.

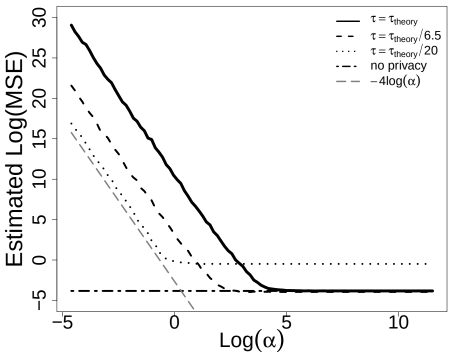
A.1 Proof of Proposition 1

Let $\hat{I}_n(\omega) = \hat{I}_n^Z(\omega) - \frac{4\tau_n^2}{\pi\alpha^2}$ with $\hat{I}_n^Z(\omega) = \frac{1}{2\pi n} \left|\sum_{t=1}^n Z_t \exp(-\mathtt{i}t\omega)\right|^2$. Then,

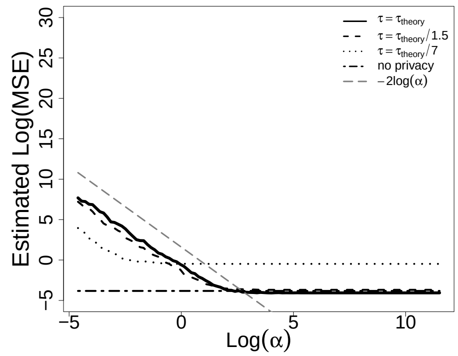
$$\hat{I}_n^Z(\omega)=\frac{1}{2\pi n}\left|\sum_{t=1}^n Z_t \exp(-\mathtt{i}t\omega)\right|^2=\frac{1}{2\pi n}\sum_{s=1}^n\sum_{t=1}^n Z_s Z_t \exp(\mathtt{i}\omega(s-t))=\frac{1}{2\pi}\sum_{|h|<n}\hat{\gamma}(h)\exp(-\mathtt{i}h\omega),$$

where $\hat{\gamma}(h) = \frac{1}{n}\sum_{t=1}^{n-|h|} Z_t Z_{t+|h|}$ is the sample covariance estimator. Thus, for $j\in\{-R, ..., R\}$,

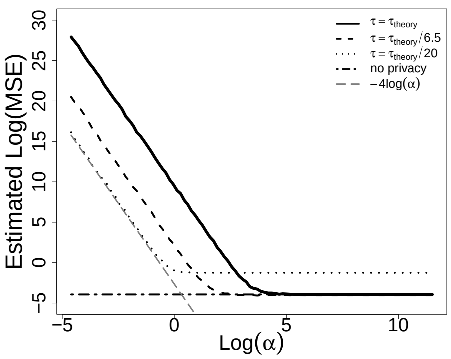
$$\hat{\sigma}_j^{\mathrm{NI}} = \int_{-\pi}^{\pi} \hat{I}_n(\omega)\exp(\mathtt{i}j\omega)\,\mathrm{d}\omega = \int_{-\pi}^{\pi} \hat{I}_n^Z(\omega)\exp(\mathtt{i}j\omega)\,\mathrm{d}\omega - \frac{8\tau_n^2}{\alpha^2}\delta_{j,0}$$



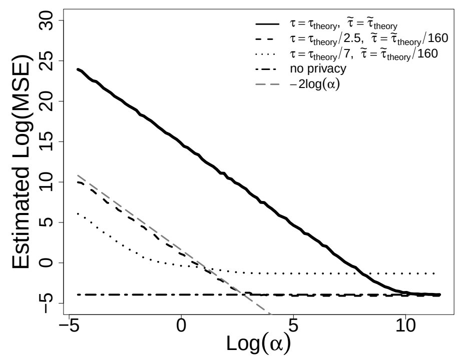
(a) Estimation of σ_0=1.44 from NI-privatized data.



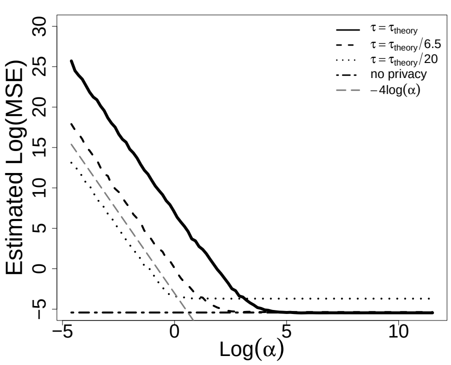
(b) Estimation of σ_0=1.44 from SI-privatized data.



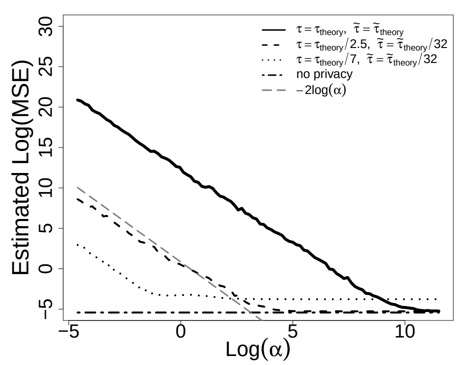
(c) Estimation of σ_2=0.92 from NI-privatized data.



(d) Estimation of σ_2=0.92 from SI-privatized data.



(e) Estimation of $f(\pi/5)$=0.24 from NI-privatized data.



(f) Estimation of $f(\pi/5)$=0.24 from SI-privatized data.

Figure 1: Example (1): Log-log plot of the estimated mean squared error (MSE) versus the privacy level α for estimation of variance (top), covariance coefficient (center) and pointwise spectral density (bottom) using NI-privatized data (left) and SI-privatized data (right). Gray dashed lines indicate the theoretical α-scaling of the MSE for NI- and SI-based estimators, including $\log(n)$-factors.

$$
\begin{aligned}
&= \int_{-\pi}^{\pi} \frac{1}{2\pi} \sum_{|h|<n} \hat{\gamma}(h) \exp(-\mathtt{i}h\omega) \exp(\mathtt{i}j\omega) \,\mathrm{d}\omega - \frac{8\tau_n^2}{\alpha^2}\delta_{j,0} \\
&= \frac{1}{2\pi} \sum_{|h|<n} \hat{\gamma}(h) \int_{-\pi}^{\pi} \exp\{\mathtt{i}\omega(j-h)\} \,\mathrm{d}\omega - \frac{8\tau_n^2}{\alpha^2}\delta_{j,0} \\
&= \sum_{|h|<n} \hat{\gamma}(h)\delta_{h,j} - \frac{8\tau_n^2}{\alpha^2}\delta_{j,0} = \begin{cases} \frac{1}{n}\sum_{t=1}^{n} Z_t^2 - \frac{8\tau_n^2}{\alpha^2}, & j=0, \\ \frac{1}{n}\sum_{t=1}^{n-|j|} Z_t Z_{t+|j|}, & j \neq 0. \end{cases}
\end{aligned} \tag{12}
$$

This implies

$$
\hat{f}_m^{\mathrm{NI}}(\omega) = \frac{1}{2\pi} \sum_{j=-m}^{m} \hat{\sigma}_j^{\mathrm{NI}} \exp(-\mathtt{i}j\omega) = \frac{1}{2\pi} \sum_{j=-m}^{m} \hat{\gamma}(j) \exp(-\mathtt{i}j\omega) - \frac{4\tau_n^2}{\pi\alpha^2}.
$$

Proof of i): As $\hat{\sigma}_j^{\mathrm{NI}} = \hat{\sigma}_{-j}^{\mathrm{NI}}$, it is sufficient to consider $j = 0, 1, ..., R$. We start with $j \neq 0$. Consider the event A_{NI} and its complement defined by

$$
A_{\mathrm{NI}} = \bigcap_{i=1}^{n} \{\widetilde{X}_i = X_i\}, \quad A_{\mathrm{NI}}^c = \bigcup_{i=1}^{n} \{\widetilde{X}_i \neq X_i\}.
$$

L_2 risk on the event A_{NI}. If the event A_{NI} occurs, then $\hat{\sigma}_j^{\mathrm{NI}}$ coincides with the untruncated estimator $\tilde{\sigma}_j^{\mathrm{NI}}$, which is defined by

$$
\tilde{\sigma}_j^{\mathrm{NI}} = \frac{1}{n} \sum_{i=1}^{n-j} (X_i + \xi_i) \cdot (X_{i+j} + \xi_{i+j}) \,. \tag{13}
$$

In particular,

$$
\mathbb{E}[(\hat{\sigma}_j^{\mathrm{NI}} - \sigma_j)^2] = \mathbb{E}[\mathbb{I}_{A_{\mathrm{NI}}}(\hat{\sigma}_j^{\mathrm{NI}} - \sigma_j)^2] + \mathbb{E}[\mathbb{I}_{A_{\mathrm{NI}}^c}(\hat{\sigma}_j^{\mathrm{NI}} - \sigma_j)^2]
$$

$$
\text{and} \quad \mathbb{E}[\mathbb{I}_{A_{\mathrm{NI}}}(\hat{\sigma}_j^{\mathrm{NI}} - \sigma_j)^2] \leq \mathbb{E}[(\tilde{\sigma}_j^{\mathrm{NI}} - \sigma_j)^2] = \mathbb{E}[(\tilde{\sigma}_j^{\mathrm{NI}} - \mathbb{E}[\tilde{\sigma}_j^{\mathrm{NI}}])^2] + (\mathbb{E}[\tilde{\sigma}_j^{\mathrm{NI}}] - \sigma_j)^2.
$$

It is straightforward to see that

$$
\mathbb{E}[\tilde{\sigma}_j^{\mathrm{NI}}] = \frac{1}{n} \sum_{i=1}^{n-j} \mathbb{E}\left[X_i X_{i+j}\right] = \frac{1}{n} \sum_{i=1}^{n-j} \sigma_j = \frac{n-j}{n}\sigma_j.
$$

As $0 \leq j \leq R < n$ and $R = \mathcal{O}(n^{1/2})$, it follows $(\mathbb{E}[\tilde{\sigma}_j^{\mathrm{NI}}] - \sigma_j)^2 = \frac{j^2}{n^2}\sigma_j^2 = \mathcal{O}(n^{-1})$. The hidden constant does not depend on j since $\sigma_j^2 \leq M_1$ by assumption. Let $\bar{\sigma}_j = \frac{1}{n}\sum_{i=1}^{n-j} X_i X_{i+j}$ be the biased estimator built from the original data. Then,

$$
\begin{aligned}
\tilde{\sigma}_j^{\mathrm{NI}} &= \frac{1}{n} \sum_{i=1}^{n-j} (X_i + \xi_i) \cdot (X_{i+j} + \xi_{i+j}) \\
&= \bar{\sigma}_j + \frac{1}{n} \sum_{i=1}^{n-j} X_i \xi_{i+j} + \frac{1}{n} \sum_{i=1}^{n-j} X_{i+j} \xi_i + \frac{1}{n} \sum_{i=1}^{n-j} \xi_{i+j} \xi_i
\end{aligned}
$$

$$=: \bar{\sigma}_j + B + C + D.$$

Since all terms of B, C and D are centered, uncorrelated and uncorrelated to the terms in $\bar{\sigma}_j$, we have

$$\mathbb{E}\left[(\tilde{\sigma}_j^{\mathrm{NI}} - \mathbb{E}[\tilde{\sigma}_j^{\mathrm{NI}}])^2\right] = \mathrm{Var}[\bar{\sigma}_j] + \mathbb{E}[B^2] + \mathbb{E}[C^2] + \mathbb{E}[D^2]. \tag{14}$$

For the first term on the right-hand side of the above equality, we obtain

$$\mathrm{Var}[\bar{\sigma}_j] = \frac{1}{n^2}\sum_{i=1}^{n-j}\sum_{k=1}^{n-j} \mathrm{Cov}\left(X_i X_{i+j}, X_k X_{k+j}\right) = \frac{1}{n^2}\sum_{i=1}^{n-j}\sum_{k=1}^{n-j} \mathbb{E}\left[X_i X_{i+j} X_k X_{k+j}\right] - \sigma_j^2.$$

By Wick's formula, we have

$$\begin{aligned}\mathbb{E}\left[X_i X_{i+j} X_k X_{k+j}\right] &= \mathbb{E}\left[X_i X_{i+j}\right]\mathbb{E}[X_k X_{k+j}] + \mathbb{E}\left[X_i X_k\right]\mathbb{E}[X_{i+j} X_{k+j}] + \mathbb{E}\left[X_i X_{k+j}\right]\mathbb{E}[X_k X_{i+j}] \\ &= \sigma_j^2 + \sigma_{|i-k|}^2 + \sigma_{|i-k-j|}\sigma_{|i+j-k|}. \end{aligned} \tag{15}$$

Using the Cauchy-Schwarz inequality and $\sum_{j\in\mathbb{Z}} \sigma_j^2 \leq M_1$, we get

$$\begin{aligned}\mathrm{Var}[\bar{\sigma}_j] &= \frac{1}{n^2}\sum_{i=1}^{n-j}\sum_{k=1}^{n-j} \sigma_{|i-k|}^2 + \sigma_{|i-k+j|}\sigma_{|i-k-j|} = \frac{1}{n^2}\sum_{h=-(n-j-1)}^{n-j-1} (n-j-|h|)(\sigma_{|h|}^2 + \sigma_{|h+j|}\sigma_{|h-j|}) \\ &\leq \frac{1}{n}\left(\sum_{h=-n}^{n} \sigma_{|h|}^2 + |\sigma_{|h+j|}||\sigma_{|h-j|}|\right) \leq \frac{2M_1}{n}.\end{aligned}$$

Next, since the ξ_i are independent of the X_i and centered,

$$\mathbb{E}[B^2] = \frac{1}{n^2}\sum_{i=1}^{n-j}\sum_{k=1}^{n-j} \mathbb{E}[X_i X_k]\mathbb{E}[\xi_{i+j}\xi_{k+j}] = \frac{8\sigma_0\tau_n^2(n-j)}{\alpha^2 n^2} \leq c\frac{8\sigma_0\tau_n^2}{\alpha^2 n}.$$

Similar computations give the same bound for $\mathbb{E}[C^2]$. For the next term in (14) we have

$$\mathbb{E}[D^2] = \frac{1}{n^2}\sum_{i=1}^{n-j}\mathbb{E}[\xi_{i+j}^2\xi_i^2] = \frac{1}{n^2}\sum_{i=1}^{n-j}\mathbb{E}[\xi_{i+j}^2]\mathbb{E}[\xi_i^2] = \frac{64\tau_n^4(n-j)}{\alpha^4 n^2} \leq c\frac{64\tau_n^4}{\alpha^4 n}.$$

Inserting the previous bounds into (14), we get, for $j \in \{1, 2, ..., R\}$,

$$\mathbb{E}[(\hat{\sigma}_j^{\mathrm{NI}} - \sigma_j)^2] \lesssim \frac{2M_1}{n} + \frac{8\sigma_0\tau_n^2}{\alpha^2 n} + \frac{64\tau_n^4}{\alpha^4 n} + \mathbb{E}[\mathbb{I}_{A_{\mathrm{NI}}^c}(\hat{\sigma}_j - \sigma_j)^2].$$

If $j = 0$, then define $\tilde{\sigma}_0^{\mathrm{NI}}$ by

$$\tilde{\sigma}_0^{\mathrm{NI}} = \frac{1}{n}\sum_{i=1}^{n}(X_i + \xi_i)^2 - \frac{8\tau_n^2}{\alpha^2}. \tag{16}$$

As $\mathbb{E}[\tilde{\sigma}_0^{\mathrm{NI}}] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[X_i^2\right] + \mathbb{E}[\xi_i^2] - \frac{8\tau_n^2}{\alpha^2} = \sigma_0$, it follows

$$\mathbb{E}[\mathbb{I}_{A_{\mathrm{NI}}}(\hat{\sigma}_0^{\mathrm{NI}} - \sigma_0)^2] \leq \mathbb{E}[(\tilde{\sigma}_0^{\mathrm{NI}} - \sigma_0)^2] = \mathrm{Var}(\bar{\sigma}_0) + 2\mathbb{E}[C^2] + \mathbb{E}[D^2]$$

with $C = \frac{2}{n}\sum_{i=1}^n X_i\xi_i$ and $D = \frac{1}{n}\sum_{i=1}^n \xi_i^2$. Similar calculations as before yield

$$\mathbb{E}[(\hat{\sigma}_0^{\mathrm{NI}} - \sigma_0)^2] \lesssim \frac{2M_1}{n} + \frac{8\sigma_0\tau_n^2}{\alpha^2 n} + \frac{96\tau_n^4}{\alpha^4 n} + \mathbb{E}[\mathbb{I}_{A_{\mathrm{NI}}^c}(\hat{\sigma}_0^{\mathrm{NI}} - \sigma_0)^2].$$

L_2 risk on the event A_{NI}^c. By Parseval's equality and the results of Kroll (2024) (see proof of Theorem 3.2 on page 19), it follows with $\tau_n^2 = 56\log^{1+\delta}(n)$ for some $c, \delta > 0$, that

$$\mathbb{E}[|\hat{\sigma}_j^{\mathrm{NI}} - \sigma_j|^2\mathbb{I}_{A_{\mathrm{NI}}^c}] \le \mathbb{E}[\|\hat{f}_m^{\mathrm{NI}} - f_m\|_2^2\mathbb{I}_{A_{\mathrm{NI}}^c}] \le c\left(\frac{\tau_n^4}{n} \vee \frac{\tau_n^4}{n\alpha^4}\right), \quad j \in \{0, 1, ..., R\},$$

where $f_m(\cdot) = \frac{1}{2\pi}\sum_{k=-m}^m \sigma_k \exp(-\mathtt{i}k\cdot)$ and $R \le m \le n$. Then, for sufficiently large n,

$$\mathbb{E}|\hat{\sigma}_j^{\mathrm{NI}} - \sigma_j|^2 \le c\left[\frac{\log^{2+2\delta}(n)}{n} \vee \frac{\log^{2+2\delta}(n)}{n\alpha^4}\right].$$

Proof of ii) and iii): Let $\omega \in [-\pi, \pi]$ be arbitrary but fixed. Consider the event A_{NI} and its complement defined as before.

L_2 risk on the event A_{NI}. If the event A_{NI} occurs, then $\hat{f}_m^{\mathrm{NI}}(\omega)$ coincides with the untruncated estimator $\tilde{f}_m^{\mathrm{NI}}(\omega)$, which is defined by

$$\begin{aligned}\tilde{f}_m^{\mathrm{NI}}(\omega) &= \frac{1}{2\pi}\sum_{j=-m}^m \tilde{\sigma}_j^{\mathrm{NI}}\exp(-\mathtt{i}j\omega)\\ &= \frac{1}{2\pi n}\sum_{j=-m}^m\sum_{i=1}^{n-|j|}\left[(X_i + \xi_i)\cdot(X_{i+|j|} + \xi_{i+|j|})\exp(-\mathtt{i}j\omega) - \delta_{j,0}\frac{8\tau_n^2}{\alpha^2}\right],\end{aligned}$$

where we used identity (12), (13) and (16). In particular,

$$\mathbb{E}[\{\hat{f}_m^{\mathrm{NI}}(\omega) - f(\omega)\}^2] = \mathbb{E}[\mathbb{I}_{A_{\mathrm{NI}}}\{\hat{f}_m^{\mathrm{NI}}(\omega) - f(\omega)\}^2] + \mathbb{E}[\mathbb{I}_{A_{\mathrm{NI}}^c}\{\hat{f}_m^{\mathrm{NI}}(\omega) - f(\omega)\}^2]$$

and

$$\begin{aligned}\mathbb{E}[\mathbb{I}_{A_{\mathrm{NI}}}\{\hat{f}_m^{\mathrm{NI}}(\omega) - f(\omega)\}^2] &\le \mathbb{E}[\{\tilde{f}_m^{\mathrm{NI}}(\omega) - f(\omega)\}^2]\\ &= \mathbb{E}[\{\tilde{f}_m^{\mathrm{NI}}(\omega) - \mathbb{E}[\tilde{f}_m^{\mathrm{NI}}(\omega)]\}^2] + \{\mathbb{E}[\tilde{f}_m^{\mathrm{NI}}(\omega)] - f(\omega)\}^2.\end{aligned}$$

We start with calculating the expectation of $\tilde{f}_m^{\mathrm{NI}}(\omega)$.

$$\begin{aligned}\mathbb{E}[\tilde{f}_m^{\mathrm{NI}}(\omega)] &= \frac{1}{2\pi n}\sum_{j=-m}^m\sum_{i=1}^{n-|j|}\left[\left(\mathbb{E}[X_iX_{i+|j|}] + \mathbb{E}[\xi_i\xi_{i+|j|}]\right)\exp(-\mathtt{i}j\omega) - \delta_{j,0}\frac{8\tau_n^2}{\alpha^2}\right]\\ &= \sum_{j=-m}^m \frac{n-|j|}{2\pi n}\sigma_j\exp(-\mathtt{i}j\omega) = \frac{1}{2\pi}\sum_{|j|\le m} a_j\sigma_j\exp(-\mathtt{i}j\omega)\end{aligned}$$

with $a_j = 1 - |j|/n$ for $j = -m, ..., m$. Then, using (2),

$$\left|\frac{1}{2\pi}\sum_{|j|\le m} a_j\sigma_j\exp(-\mathtt{i}j\omega) - f(\omega)\right|$$

$$= \frac{1}{2\pi}\left|\sum_{|j|\le m} -\frac{|j|}{n}\sigma_j \exp(-\mathtt{i}j\omega) - \sum_{|j|>m}\sigma_j \exp(-\mathtt{i}j\omega)\right|$$
$$\le \frac{Mm}{2\pi n} + \frac{1}{2\pi}\left|\sum_{|j|>m}\sigma_j \exp(-\mathtt{i}j\omega)\right|.$$

If $f \in W^{s,2}(L)$, $s > 1/2$, then $\sum_{j\in\mathbb{Z}} |\sigma_j|^2|j|^{2s} < L$. Thus, by the Cauchy-Schwarz inequality we obtain

$$\left|\sum_{|j|>m}\sigma_j \exp(-\mathtt{i}j\omega)\right| \le c\sum_{|j|>m}|\sigma_j|\frac{|j|^s}{|j|^s} \le c\sum_{|j|>m}|\sigma_j|^2|j|^{2s}\sum_{|j|>m}\frac{1}{|j|^{2s}} = \mathcal{O}(m^{-s+1/2}). \tag{17}$$

If $f \in W^{s,\infty}(L_0, L)$, then by the approximation results of Zygmund (2002, page 120)

$$\left|\frac{1}{2\pi}\sum_{|j|>m}\sigma_j \exp(-\mathtt{i}j\omega)\right| = \mathcal{O}\left(\frac{\log(m)}{m^s}\right). \tag{18}$$

Thus, we obtain for the squared bias

$$\{\mathbb{E}[\tilde{f}_m^{\mathrm{NI}}(\omega)] - f(\omega)\}^2 \le C\frac{m^2}{n^2} + C\cdot\begin{cases}\log(m)^2 m^{-2s}, & f\in W^{s,\infty}(L_0,L),\\ m^{-2s+1}, & f\in W^{s,2}(L),\end{cases}$$

where the constant $C > 0$ depends on s, L, if $f \in W^{s,2}(L)$, and additionally on L_0, if $f \in W^{s,\infty}(L_0, L)$.

To analyze the variance, define the estimator without truncation or added Laplace noise by

$$\bar{f}_m(\omega) = \frac{1}{2\pi n}\sum_{j=-m}^{m}\sum_{i=1}^{n-|j|} X_i X_{i+|j|}\exp(-\mathtt{i}j\omega).$$

Then,

$$\begin{aligned}
\tilde{f}_m^{\mathrm{NI}}(\omega) =& \frac{1}{2\pi n}\sum_{j=-m}^{m}\sum_{i=1}^{n-|j|}\left[(X_i+\xi_i)\cdot(X_{i+|j|}+\xi_{i+|j|})\exp(-\mathtt{i}j\omega) - \delta_{j,0}\frac{8\tau_n^2}{\alpha^2}\right]\\
=& \frac{1}{2\pi n}\sum_{j=-m}^{m}\sum_{i=1}^{n-|j|} X_iX_{i+|j|}\exp(-\mathtt{i}j\omega) + \frac{1}{2\pi n}\sum_{j=-m}^{m}\sum_{i=1}^{n-|j|} X_i\xi_{i+|j|}\exp(-\mathtt{i}j\omega)\\
&+ \frac{1}{2\pi n}\sum_{j=-m}^{m}\sum_{i=1}^{n-|j|} \xi_iX_{i+|j|}\exp(-\mathtt{i}j\omega) + \frac{1}{2\pi n}\sum_{j=-m}^{m}\sum_{i=1}^{n-|j|} \xi_i\xi_{i+|j|}\exp(-\mathtt{i}j\omega) - \delta_{j,0}\frac{8\tau_n^2}{\alpha^2}\\
=:& \bar{f}_m(\omega) + B + C + D.
\end{aligned}$$

In particular, $\mathbb{E}[\tilde{f}_m^{\mathrm{NI}}(\omega)] = \mathbb{E}[\bar{f}_m(\omega)]$ and

$$\mathbb{E}[\{\tilde{f}_m^{\mathrm{NI}}(\omega) - \mathbb{E}[\tilde{f}_m^{\mathrm{NI}}(\omega)]\}^2] = \mathrm{Var}[\bar{f}_m(\omega)] + \mathbb{E}[B^2] + \mathbb{E}[C^2] + \mathbb{E}[D^2]. \tag{19}$$

Using $\bar{\sigma}_j = \frac{1}{n}\sum_{i=1}^{n-|j|} X_i X_{i+|j|}$, we have

$$
\begin{aligned}
&(2\pi n)^2 \mathrm{Var}[\bar{f}_m(\omega)] \\
&= n^2 \sum_{l=-m}^{m}\sum_{k=-m}^{m} \exp\{-\mathtt{i}\omega(l-k)\}\mathrm{Cov}(\bar{\sigma}_l, \bar{\sigma}_k) \\
&= \sum_{l=-m}^{m}\sum_{k=-m}^{m} \exp\{-\mathtt{i}\omega(l-k)\} \sum_{r=1}^{n-|l|}\sum_{s=1}^{n-|k|} \mathrm{Cov}(X_r X_{r+|l|}, X_s X_{s+|k|}).
\end{aligned} \tag{20}
$$

Now,

$$
\begin{aligned}
&\mathrm{Cov}(X_r X_{r+|l|}, X_s X_{s+|k|}) \\
&= \mathbb{E}[X_r X_{r+|l|}]\mathbb{E}[X_s X_{s+|k|}] + \mathbb{E}[X_r X_s]\mathbb{E}[X_{s+|k|} X_{r+|l|}] + \mathbb{E}[X_r X_{s+|k|}]\mathbb{E}[X_s X_{r+|l|}] - \sigma_{|l|}\sigma_{|k|} \\
&= \sigma_{|r-s|}\sigma_{|r-s-|k|+|l||} + \sigma_{|r-s-|k||}\sigma_{|r-s+|l||}.
\end{aligned}
$$

Substituting this into (20), we get

$$
\begin{aligned}
&(2\pi n)^2 \mathrm{Var}[\bar{f}_m(\omega)] \\
&\leq \sum_{l=-m}^{m}\sum_{k=-m}^{m}\sum_{h=-(n-1)}^{n-1} (n-1-|h|) \left| \sigma_{|h|}\sigma_{|h-|k|+|l||} + \sigma_{|h-|k||}\sigma_{|h+|l||} \right| \\
&\leq \sum_{l=-m}^{m}\sum_{h=-(n-1)}^{n-1} (n-1-|h|) \left[\left|\sigma_{|h|}\right| \sum_{k=-m}^{m} \left|\sigma_{|h-|k|+|l||}\right| + \left|\sigma_{|h+|l||}\right| \sum_{k=-m}^{m} \left|\sigma_{|h-|k||}\right| \right] \\
&\leq \sum_{l=-m}^{m}\sum_{h=-(n-1)}^{n-1} (n-1-|h|)(|\sigma_{|h|}| + |\sigma_{|h+|l||}|)M.
\end{aligned}
$$

Since $\sum_{h=-(n-1)}^{n-1}\left(1-\frac{1+|h|}{n}\right)(|\sigma_{|h|}| + |\sigma_{|h+|l||}|) \leq 2M$, it follows $\mathrm{Var}[\bar{f}_m(\omega)] \leq Cmn^{-1}$. Similar computations as in the proof of Theorem 5 show that $\mathbb{E}[B^2]$ and $\mathbb{E}[C^2]$ can be bounded by $\frac{\tau_n^2 m}{n\alpha^2}$ up to constants. For the last term in Equation (19) we have

$$
\mathbb{E}[D^2] = \mathrm{Var}[D] = \frac{1}{4\pi^2 n^2}\sum_{j=-m}^{m}\sum_{i=1}^{n-|j|} \mathrm{Var}[\xi_i \xi_{i+|j|}] \leq c\frac{m\tau_n^4}{\alpha^4 n}.
$$

Summing up, we get

$$
\begin{aligned}
\mathbb{E}[\{\tilde{f}_m^{\mathrm{NI}}(\omega) - f(\omega)]\}^2] = &\mathbb{E}[\mathbb{I}_{A_{\mathrm{NI}}^c}\{\hat{f}_m^{\mathrm{NI}}(\omega) - f(\omega)\}^2] + \mathcal{O}\left(\frac{m}{n} \vee \frac{m\tau_n^4}{n\alpha^4}\right) \\
&+ \begin{cases} \mathcal{O}(\log(m)^2 m^{-2s}), & f \in W^{s,\infty}(L_0, L), \\ \mathcal{O}(m^{-2s+1}), & f \in W^{s,2}(L). \end{cases}
\end{aligned} \tag{21}
$$

L_2 risk on the event A_{NI}. Application of the Cauchy-Schwarz inequality gives

$$
\mathbb{E}[\mathbb{I}_{A_{\mathrm{NI}}^c}\{\hat{f}_m^{\mathrm{NI}}(\omega) - f(\omega)\}^2] \leq \left(\mathbb{E}[\{\hat{f}_m^{\mathrm{NI}}(\omega) - f(\omega)\}^4] \cdot \Pr[A_{\mathrm{NI}}^c]\right)^{1/2}.
$$

From the proof of Theorem 3.2. of Kroll (2024) follows

$$\begin{aligned}\mathbb{E}[\{\hat{f}_m^{\mathrm{NI}}(\omega) - f(\omega)\}^4] &\leq \mathbb{E}[\|\hat{f}_m^{\mathrm{NI}} - f\|_\infty^4] \\ &\leq \mathbb{E}[\|\hat{f}_m^{\mathrm{NI}} - f_m\|_\infty^4] + \|f_m - f\|_\infty^4 \\ &\leq \left(M + \frac{\tau_n^2}{\alpha^2}\right)^4 + \frac{n^4\tau_n^8}{1 \wedge \alpha^8} + \left(\sum_{|l|>m} |\sigma_l|\right)^4.\end{aligned}$$

Furthermore, it is shown by Kroll (2024) that if $\tau_n^2 = 56\log^{1+\delta}(n)$, then $\Pr(A_{\mathrm{NI}}^c) \leq n^{-6}$. Then,

$$\mathbb{E}[\mathbb{I}_{A_{\mathrm{NI}}^c}\{\hat{f}_m^{\mathrm{NI}}(\omega) - f(\omega)\}^2] \leq \left(\frac{\tau_n^4}{n} \vee \frac{\tau_n^4}{n\alpha^4}\right). \tag{22}$$

Combining (21) and (22), setting $\tau_n^2 = 56\log^{1+\delta}(n)$ gives

$$\mathbb{E}|\hat{f}_m^{\mathrm{NI}}(\omega) - f(\omega)|^2 = \mathcal{O}\left(\frac{\tau_n^4}{n}\right) + \mathcal{O}\left(\frac{m \vee m\tau_n^4\alpha^{-4}}{n}\right) + \begin{cases}\mathcal{O}(\log(m)m^{-2s}), & f\in W^{s,\infty}(L_0, L), \\ \mathcal{O}(m^{-2s+1}), & f\in W^{s,2}(L).\end{cases}$$

Optimizing $m^{-2s} + m \cdot (1 \vee \tau_n^4\alpha^{-4})/n$ with respect to m gives

$$m = \left(\frac{1}{n} \vee \frac{\tau_n^4}{n\alpha^4}\right)^{-\frac{1}{2s+1}}.$$

Analogously, optimizing $m^{-2s+1} + m \cdot (1 \vee \tau_n^4\alpha^{-4})/n$ with respect to m gives

$$m = \left(\frac{1}{n} \vee \frac{\tau_n^4}{n\alpha^4}\right)^{-\frac{1}{2s}}.$$

As $f \in W^{s,\infty}(L_0, L)$ resp. $f \in W^{s,2}(L)$ was chosen arbitrary, we obtain

$$\sup_{f\in W^{s,\infty}(L_0,L)} \mathbb{E}|\hat{f}_m^{\mathrm{NI}}(\omega) - f(\omega)|^2 \leq c_1 \left\{\frac{\log(n)}{n} \vee \frac{\log^{3+2\delta}(n)}{n\alpha^4}\right\}^{\frac{2s}{2s+1}} \vee \frac{\log^{2+2\delta}(n)}{n}$$

and

$$\sup_{f\in W^{s,2}(L)} \mathbb{E}|\hat{f}_m^{\mathrm{NI}}(\omega) - f(\omega)|^2 \leq c_2 \left\{\frac{1}{n} \vee \frac{\log^{2+2\delta}(n)}{n\alpha^4}\right\}^{\frac{2s-1}{2s}} \vee \frac{\log^{2+2\delta}(n)}{n}.$$

∎

A.2 Proof of Lemma 2

First consider j in $\{1, ..., n-1\}$. It is sufficient to check that for all $i = 1, ..., n$, for all z and $\bar{z}$ and for all x different from x' the ratio of conditional density functions q of Q is bounded by e^α, that is,

$$\frac{q^{Z_i|X_i=x,Z_1,...,Z_{i-1}}(z)}{q^{Z_i|X_i=x',Z_1,...,Z_{i-1}}(z)} \leq e^\alpha, \quad \text{for } i \leq j, \text{ and}$$

$$\frac{q^{(Z_i,\bar{Z}_{i,j})|X_i=x,(Z_1,\bar{Z}_{1,j}),\dots,(Z_{i-1},\bar{Z}_{i-1,j})}(z,\bar{z})}{q^{(Z_i,\bar{Z}_{i,j})|X_i=x',(Z_1,\bar{Z}_{1,j}),\dots,(Z_{i-1},\bar{Z}_{i-1,j})}(z,\bar{z})} \le e^{\alpha}, \quad \text{for } i \ge j+1.$$

By definition of the privacy mechanism described in (4) and (5), the conditional density of $(Z_i, \bar{Z}_{i,j})$ given $X_i = x$ and all the previously built variables is

$$\begin{aligned}
&q^{Z_i|X_i=x,Z_1,\dots,Z_{i-1}}(z) = \frac{\alpha}{8\tau_n}\exp\left(-\frac{\alpha}{4\tau_n}|z-\tilde{x}|\right), \quad \text{for } i \le j, \text{ and} \\
&q^{(Z_i,\bar{Z}_{i,j})|X_i=x,(Z_1,\bar{Z}_{1,j}),\dots,(Z_{i-1},\bar{Z}_{i-1,j})}(z,\bar{z}) \\
&= \frac{\alpha}{8\tau_n}\exp\left(-\frac{\alpha}{4\tau_n}|z-\tilde{x}|\right)\cdot\frac{\alpha}{8\tilde{\tau}_n}\exp\left(-\frac{\alpha}{4\tilde{\tau}_n}|\bar{z}-\tilde{U}_i(x)|\right), \quad \text{for } i \ge j+1,
\end{aligned}$$

where $\widetilde{X}$ denotes x trimmed at $\pm\tau_n$ and $\tilde{U}_i(x)$ denotes xZ_{i-j} trimmed at $\pm\tilde{\tau}_n$. Thus, the likelihood ratio above can be treated simultaneously by

$$\begin{aligned}
&= \exp\left\{-\frac{\alpha}{4\tau_n}|z-\tilde{x}| + \frac{\alpha}{4\tau_n}|z-\tilde{x}'| - \left(\frac{\alpha}{4\tilde{\tau}_n}|\bar{z}-\tilde{U}_i(x)| - \frac{\alpha}{4\tilde{\tau}_n}|\bar{z}-\tilde{U}_i(x')|\right)\cdot\mathbf{1}(i \ge j+1)\right\} \\
&\le \exp\left(\frac{\alpha}{4\tau_n}|\tilde{x}-\tilde{x}'| + \frac{\alpha}{4\tilde{\tau}_n}|\tilde{U}_i(x)-\tilde{U}_i(x')|\right).
\end{aligned}$$

We conclude after noting that $|x-\tilde{x}'| \le 2\tau_n$ and that $|\tilde{U}_i(x)-\tilde{U}_i(x')| \le 2\tilde{\tau}_n$ by construction.

If $j = 0$, then for all $i = 1, \dots, n$, for all $\bar{z}$ and for all x different from x', we have

$$\frac{q^{(\bar{Z}_{i,0})|X_i=x}(\bar{z})}{q^{(\bar{Z}_{i,0})|X_i=x'}(\bar{z})} = \exp\left\{-\frac{\alpha}{2\tau_n}|\bar{z}-\tilde{U}_i(x)| + \frac{\alpha}{2\tau_n}|\bar{z}-\tilde{U}_i(x')|\right\} \le \exp\left(\frac{\alpha}{2\tau_n}|\tilde{U}_i(x)-\tilde{U}_i(x')|\right),$$

where $\tilde{U}_i(x)$ denotes x^2 trimmed at $\pm\tilde{\tau}_n$. We conclude after noting that $|\tilde{U}_i(x)-\tilde{U}_i(x')| \le 2\tau_n$ by construction. ∎

A.3 Proof of Theorem 3

Let $R \in \mathbb{N}$ such that $0 < R < n$ and $R = o(n)$. We start with $j \in \{1, \dots, R\}$. Consider the event A and its complement defined by

$$A = \bigcap_{i=1}^{n}\{\widetilde{X}_i = X_i \text{ and } (\widetilde{W}_{i,j} = W_{i,j})\mathbf{1}(i > j)\}, \quad A^c = \bigcup_{i=1}^{n}\{\widetilde{X}_i \ne X_i \text{ or } (\widetilde{W}_{i,j} \ne W_{i,j})\mathbf{1}(i > j)\}.$$

L_2 risk on the event A. If the event A occurs, then $\hat{\sigma}_j$ coincides with the untruncated estimator $\tilde{\sigma}_j$, which is defined by

$$\tilde{\sigma}_j = \frac{1}{n-j}\sum_{i=j+1}^{n} X_i\cdot(X_{i-j}+\xi_{i-j}) + \tilde{\xi}_i.$$

In particular,

$$\mathbb{E}[(\hat{\sigma}_j-\sigma_j)^2] = \mathbb{E}[\mathbb{I}_A(\hat{\sigma}_j-\sigma_j)^2] + \mathbb{E}[\mathbb{I}_{A^c}(\hat{\sigma}_j-\sigma_j)^2]$$

$$\text{and} \quad \mathbb{E}[\mathbb{I}_A(\hat{\sigma}_j - \sigma_j)^2] \leq \mathbb{E}[(\tilde{\sigma}_j - \sigma_j)^2] = \mathbb{E}[(\tilde{\sigma}_j - \mathbb{E}[\tilde{\sigma}_j])^2] + (\mathbb{E}[\tilde{\sigma}_j] - \sigma_j)^2.$$

It is straightforward to see that

$$\mathbb{E}[\tilde{\sigma}_j] = \frac{1}{n-j} \sum_{i=j+1}^{n} \mathbb{E}\left[X_i X_{i-j}\right] = \frac{1}{n-j} \sum_{i=j+1}^{n} \sigma_j = \sigma_j.$$

Let $\bar{\sigma}_j = \frac{1}{n-j} \sum_{i=j+1}^{n} X_i X_{i-j}$ be the unbiased estimator built using the data. Then,

$$\begin{aligned}\tilde{\sigma}_j &= \frac{1}{n-j} \sum_{i=j+1}^{n} X_i \cdot (X_{i-j} + \xi_{i-j}) + \tilde{\xi}_i = \bar{\sigma}_j + \frac{1}{n-j} \sum_{i=j+1}^{n} X_i \xi_{i-j} + \frac{1}{n-j} \sum_{i=j+1}^{n} \tilde{\xi}_i \\ &=: \bar{\sigma}_j + B + C.\end{aligned}$$

Since all terms of B and C are centered, uncorrelated and uncorrelated to the terms in $\bar{\sigma}_j$, we have

$$\mathbb{E}\left[(\tilde{\sigma}_j - \mathbb{E}[\tilde{\sigma}_j])^2\right] = \mathrm{Var}[\bar{\sigma}_j] + \mathbb{E}[B^2] + \mathbb{E}[C^2]. \tag{23}$$

By Wick's formula, see Equation (15), and similar computations as in the proof of Proposition 1*i*), we have

$$\mathrm{Var}[\bar{\sigma}_j] = \frac{1}{(n-j)^2} \sum_{i=j+1}^{n} \sum_{k=j+1}^{n} \mathrm{Cov}\left(X_i X_{i-j}, X_k X_{k-j}\right) \leq \frac{2}{(n-j)} \sum_{h=-n}^{n} \sigma_{|h|}^2 \leq \frac{2M_1}{(n-j)}.$$

Next, since the ξ_i are independent of the X_i and centered,

$$\mathbb{E}[B^2] = \frac{1}{(n-j)^2} \sum_{i=j+1}^{n} \sum_{k=j+1}^{n} \mathbb{E}[X_i X_k] \mathbb{E}[\xi_{i-j}\xi_{k-j}] = \frac{8\sigma_0 \tau_n^2}{\alpha^2 (n-j)}.$$

For the last term in (23) we have

$$\mathbb{E}[C^2] = \frac{1}{(n-j)^2} \sum_{i=j+1}^{n} \mathbb{E}[\tilde{\xi}_i^2] = \frac{8\tilde{\tau}_n^2}{\alpha^2(n-j)}.$$

Then,

$$\mathbb{E}[(\hat{\sigma}_j - \sigma_j)^2] \leq \frac{2M_1}{(n-j)} + \frac{8M\tau_n^2}{\alpha^2(n-j)} + \frac{8\tilde{\tau}_n^2}{\alpha^2(n-j)} + \mathbb{E}[\mathbb{I}_{A^c}(\hat{\sigma}_j - \sigma_j)^2].$$

To conclude the proof, we show that $\mathbb{E}[\mathbb{I}_{A^c}(\hat{\sigma}_j - \sigma_j)^2]$ is not larger than the other terms. *L_2 risk on the event A^c.* Application of the Cauchy-Schwarz inequality gives

$$\mathbb{E}[\mathbb{I}_{A^c}(\hat{\sigma}_j - \sigma_j)^2] \leq \left(\mathbb{E}[(\hat{\sigma}_j - \sigma_j)^4] \cdot \Pr[A^c]\right)^{1/2}.$$

Then, using $|\widetilde{W}_{i,j}| \leq \tilde{\tau}_n$, it holds for some constants $c_1, ..., c_5 > 0$, that

$$\mathbb{E}[(\hat{\sigma}_j - \sigma_j)^4] \leq c_1 \mathbb{E}[\hat{\sigma}_j^4] + c_2 \leq \frac{c_3}{(n-j)^4} \left\{ \mathbb{E}\left[\left(\sum_{i=j+1}^{n} \widetilde{W}_{i,j}\right)^4\right] + \mathbb{E}\left[\left(\sum_{i=j+1}^{n} \tilde{\xi}_i\right)^4\right] \right\} + c_2$$

$$\leq c_4\tilde{\tau}_n^4 + \frac{c_5\tilde{\tau}_n^4}{\alpha^4} + c_2 = \mathcal{O}(\tilde{\tau}_n^4 + \tilde{\tau}_n^4\alpha^{-4}).$$

Moreover,

$$\begin{aligned}
\Pr[A^c] &\leq \sum_{i=1}^n \Pr[\{\widetilde{X}_i \neq X_i \text{ or } \{\widetilde{W}_{i,j} \neq W_{i,j}\} \cdot \mathbf{1}(i > j)\}] \\
&\leq \sum_{i=1}^n \Pr[\widetilde{X}_i \neq X_i] + \Pr[\{\widetilde{X}_i = X_i \text{ and } \{\widetilde{W}_{i,j} \neq W_{i,j}\} \cdot \mathbf{1}(i > j)\}].
\end{aligned}$$

We start with the first term above and use that $\{\widetilde{X}_i \neq X_i\} = \{|X_i| > \tau_n\}$. As $X_i \sim \mathcal{N}(0, \sigma_0)$ we have

$$\sum_{i=1}^n \Pr[\widetilde{X}_i \neq X_i] \leq 2n \exp\left(-\frac{\tau_n^2}{2\sigma_0}\right), \tag{24}$$

which is of the order $\mathcal{O}(n^{-3})$ for $\tau_n^2 \geq 8\log^{1+\delta}(n)$ for some $\delta > 0$. For $i > j$, we have

$$\begin{aligned}
&\sum_{i=j+1}^n \Pr[\{\widetilde{X}_i = X_i \text{ and } \{\widetilde{W}_{i,j} \neq W_{i,j}\} \cdot \mathbf{1}(i > j)\}] \\
&\leq \sum_{i=j+1}^n \Pr\left[\left|\widetilde{X}_i \cdot Z_{i-j}\right| \geq \tilde{\tau}_n\right] \leq n \Pr\left[\left|\widetilde{X}_{i-j} + \xi_{i-j}\right| \geq \tilde{\tau}_n/\tau_n\right] \\
&\leq Cn \Pr\left[|\xi_{i-j}| \geq \tilde{\tau}_n/\tau_n\right] \leq \frac{Cn\alpha}{4\tau_n} \exp\left(-\frac{\tilde{\tau}_n\alpha}{4\tau_n^2}\right)
\end{aligned}$$

as the subexponential tail is dominating and is of the order $\mathcal{O}(n^{-3})$ if $\tilde{\tau}_n\tau_n^{-2} \geq 16\log^{1+\delta}(n)$. Setting $\tau_n^2 = 8\log^{1+\delta}(n)$ and $\tilde{\tau}_n = 16\log^{1+\delta}(n)\tau_n^2$ for some $\delta > 0$ finally gives for large n that

$$\mathbb{E}[(\hat{\sigma}_j - \sigma_j)^2] \leq c\left\{\frac{1}{n} \vee \frac{\log^{4+4\delta}(n)}{\alpha^2 n}\right\}.$$

In particular, we used that for $j \leq R$ and $R = o(n)$ holds $1/(n-j) = \mathcal{O}(n)$. Similar computations for $j = 0$. As the constants do not depend on $(\sigma_k)_{k\in\mathbb{Z}}$ but only on the global bounds $\sum_{j\in\mathbb{Z}} |\sigma_j| < M$ and $\sum_{j\in\mathbb{Z}} \sigma_j^2 \leq M_1$, we obtain

$$\sup_{\sigma:\, \|\sigma\|_2^2 \leq M_1} \mathbb{E}[(\hat{\sigma}_j - \sigma_j)^2] \leq c\left\{\frac{1}{n} \vee \frac{\log^{4+4\delta}(n)}{\alpha^2 n}\right\}.$$

■

A.4 Proof of Lemma 4

In view of Lemma 2 the case $i \leq K$ is proven. It is therefore sufficient to check that for all $i = K+1, \ldots, n$, for all z and $\tilde{z}$ and for all x different from x' the ratio of conditional density functions q of Q is bounded, that is,

$$\frac{q^{(Z_i,\widetilde{Z}_i)|X_i=x,(Z_1,\widetilde{Z}_1),\ldots,(Z_{i-1},\widetilde{Z}_{i-1})}(z,\tilde{z})}{q^{(Z_i,\widetilde{Z}_i)|X_i=x',(Z_1,\widetilde{Z}_1),\ldots,(Z_{i-1},\widetilde{Z}_{i-1})}(z,\tilde{z})} \leq e^{\alpha}.$$

By definition of the privacy mechanism described in (4) and (8), the conditional density of $(Z_i, \widetilde{Z}_i)$ given $X_i = x$ and all the previously built variables is

$$
\begin{aligned}
&q^{(Z_i,\widetilde{Z}_i)|X_i=x,(Z_1,\widetilde{Z}_1),\ldots,(Z_{i-1},\widetilde{Z}_{i-1})}(z,\tilde{z}) \\
&= \frac{\alpha}{8\tau_n}\exp\left(-\frac{\alpha}{4\tau_n}|z-\tilde{x}|\right)\cdot\frac{\alpha}{8\tilde{\tau}_n}\exp(-\frac{\alpha}{4\tilde{\tau}_n}|\tilde{z}-\tilde{V}_i(x,\omega)|), \quad \text{for } i \geq K+1,
\end{aligned}
$$

where $\widetilde{X}$ denotes x trimmed at $\pm\tau_n$ and $\tilde{V}_i(x)$ denotes $x^2 + \sum_{1\leq|k|\leq K} a_k x Z_{i-|k|}\exp(\mathtt{i}\omega k)$ trimmed at $\pm\tilde{\tau}_n$. Thus,

$$
\begin{aligned}
&\frac{q^{(Z_i,\widetilde{Z}_i)|X_i=x,(Z_1,\widetilde{Z}_1),\ldots,(Z_{i-1},\widetilde{Z}_{i-1})}(z,\tilde{z})}{q^{(Z_i,\widetilde{Z}_i)|X_i=x',(Z_1,\widetilde{Z}_1),\ldots,(Z_{i-1},\widetilde{Z}_{i-1})}(z,\tilde{z})} \\
&= \exp\left\{-\frac{\alpha}{4\tau_n}|z-\tilde{x}| + \frac{\alpha}{4\tau_n}|z-\tilde{x}'| - \left(\frac{\alpha}{4\tilde{\tau}_n}|\tilde{z}-\tilde{V}_i(x)| - \frac{\alpha}{4\tilde{\tau}_n}|\tilde{z}-\tilde{V}_i(x')|\right)\right\} \\
&\leq \exp\left(\frac{\alpha}{4\tau_n}|\tilde{x}-\tilde{x}'| + \frac{\alpha}{4\tilde{\tau}_n}|\tilde{V}_i(x)-\tilde{V}_i(x')|\right).
\end{aligned}
$$

We conclude by noting that $|x - \tilde{x}'| \leq 2\tau_n$ and that $|\tilde{V}_i(x) - \tilde{V}_i(x')| \leq 2\tilde{\tau}_n$. ■

A.5 Proof of Theorem 5

Since the proofs for the cases $f \in W^{s,\infty}(L_0, L)$ and for $f \in W^{s,2}(L)$ are identical apart from the treatment of the bias term, we handle both cases simultaneously. Let $\omega \in [-\pi, \pi]$ be arbitrary but fixed. Consider the event $\tilde{A}$ and its complement defined by

$$
\tilde{A} = \bigcap_{i=1}^{n}\{\widetilde{X}_i = X_i \text{ and } (\widetilde{V}_i = V_i)\mathbf{1}(i > K)\}, \qquad \tilde{A}^c = \bigcup_{i=1}^{n}\{\widetilde{X}_i \neq X_i \text{ or } (\widetilde{V}_i \neq V_i)\mathbf{1}(i > K)\}.
$$

L_2 risk on the event $\tilde{A}$. If the event $\tilde{A}$ occurs, then $\hat{f}_K(\omega)$ coincides with the untruncated estimator $\tilde{f}_K(\omega)$, which is defined by

$$
\tilde{f}_K(\omega) = \frac{1}{2\pi(n-K)}\sum_{j=K+1}^{n}\left(X_j^2 + \sum_{1\leq|k|\leq K} a_k X_j(X_{j-|k|} + \xi_{j-|k|})\exp(-\mathtt{i}\omega k) + \tilde{\xi}_j\right).
$$

In particular,

$$
\mathbb{E}[\{\hat{f}_K(\omega) - f(\omega)\}^2] = \mathbb{E}[\mathbb{I}_{\tilde{A}}\{\hat{f}_K(\omega) - f(\omega)\}^2] + \mathbb{E}[\mathbb{I}_{\tilde{A}^c}\{\hat{f}_K(\omega) - f(\omega)\}^2]
$$

and

$$
\begin{aligned}
\mathbb{E}[\mathbb{I}_{\tilde{A}}\{\hat{f}_K(\omega) - f(\omega)\}^2] &\leq \mathbb{E}[\{\tilde{f}_K(\omega) - f(\omega)\}^2] \\
&= \mathbb{E}[\{\tilde{f}_K(\omega) - \mathbb{E}[\tilde{f}_K(\omega)]\}^2] + \{\mathbb{E}[\tilde{f}_K(\omega)] - f(\omega)\}^2.
\end{aligned}
$$

We start with calculating the expectation of $\tilde{f}_K(\omega)$:

$$
\mathbb{E}[\tilde{f}_K(\omega)] = \frac{1}{2\pi(n-K)}\sum_{j=K+1}^{n}\left(\sum_{|k|\leq K} a_k \mathbb{E}[X_j X_{j-|k|}]\exp(-\mathtt{i}\omega k) + \mathbb{E}[\tilde{\xi}_j]\right)
$$

$$= \frac{1}{2\pi(n-K)} \sum_{j=K+1}^{n} \sum_{|k|\le K} a_k \sigma_k \exp(-\mathtt{i}\omega k) = \frac{1}{2\pi} \sum_{|k|\le K} a_k \sigma_k \exp(-\mathtt{i}\omega k).$$

As the last sum is the De la Valée-Poussin sum of f, Theorem 13.5 (see Chapter 3, Zygmund, 2002) implies

$$\left| \frac{1}{2\pi} \sum_{|k|\le K} a_k \sigma_k \exp(-\mathtt{i}\omega k) - f(\omega) \right| \le 4 \inf_{T_K \in \mathcal{T}_K} \|f - T_K\|_\infty,$$

where $\mathcal{T}_K$ is the class of trigonometric polynomials of degree less or equal than K. By Jackson's inequality (see Theorem 13.6 in Chapter 3, Zygmund, 2002), we have

$$\inf_{T_K \in \mathcal{T}_K} \|f - T_K\|_\infty \le C_s \omega(f^{s-\lfloor s \rfloor}, 2\pi/K) K^{-\lfloor s \rfloor},$$

where the constant $C_s > 0$ depends only on the smoothness s of f, $\lfloor s \rfloor$ is the largest integer strictly less than s and $\omega(f,\delta) = \max_{|t_1-t_2|\le\delta} |f(t_1) - f(t_2)|$ is the modulus of continuity. In particular, $\omega(f^{s-\lfloor s \rfloor}, 2\pi/K) \le L_0 K^{-(s-\lfloor s \rfloor)}$ and $\inf_{T_K\in\mathcal{T}_K} \|f - T_K\|_\infty \le C_s' K^{-s}$ for $f \in W^{s,\infty}(L_0, L)$. As $W^{s,2}(L)$ can be embedded in $W^{s-1/2,\infty}(\tilde{L}_0, \tilde{L})$ for some constants $\tilde{L}_0, \tilde{L} > 0$ depending on L, it follows $\inf_{T_K\in\mathcal{T}_K} \|f - T_K\|_\infty \le \tilde{C}_s K^{-(s-1/2)}$ for $f \in W^{s,2}(L)$. Thus, we obtain for the squared bias

$$\{\mathbb{E}[\tilde{f}_K(\omega)] - f(\omega)\}^2 \le C \cdot \begin{cases} K^{-2s}, & f \in W^{s,\infty}(L_0, L), \\ K^{-2s+1}, & f \in W^{s,2}(L), \end{cases}$$

where the constant $C > 0$ depends on s, L, if $f \in W^{s,2}(L)$, and additionally on L_0, if $f \in W^{s,\infty}(L_0, L)$.

To analyze the variance, define the estimator without truncation or added Laplace noise by

$$\bar{f}_K(\omega) = \frac{1}{2\pi(n-K)} \sum_{j=K+1}^{n} \sum_{|k|\le K} a_k X_j X_{j-|k|} \exp(-\mathtt{i}\omega k).$$

Then,

$$\begin{aligned} \tilde{f}_K(\omega) =& \frac{1}{2\pi(n-K)} \sum_{j=K+1}^{n} \left(X_j^2 + \sum_{1\le|k|\le K} a_k X_j (X_{j-|k|} + \xi_{j-|k|}) \exp(-\mathtt{i}\omega k) + \tilde{\xi}_j \right) \\ =& \frac{1}{2\pi(n-K)} \sum_{j=K+1}^{n} \sum_{|k|\le K} a_k X_j X_{j-|k|} \exp(-\mathtt{i}\omega k) \\ &+ \frac{1}{2\pi(n-K)} \sum_{j=K+1}^{n} \sum_{1\le|k|\le K} a_k X_j \xi_{j-|k|} \exp(-\mathtt{i}\omega k) + \frac{1}{2\pi(n-K)} \sum_{j=K+1}^{n} \tilde{\xi}_j \\ =:& \bar{f}_K(\omega) + B + C. \end{aligned}$$

Since all mixed terms have zero expectation and $\mathbb{E}[\tilde{f}_K(\omega)] = \mathbb{E}[\bar{f}_K(\omega)]$, we obtain

$$\mathbb{E}[\{\tilde{f}_K(\omega) - \mathbb{E}[\tilde{f}_K(\omega)]\}^2] = \mathrm{Var}[\bar{f}_K(\omega)] + \mathbb{E}[B^2] + \mathbb{E}[C^2]. \tag{25}$$

In particular, using $\bar{\sigma}_k = \frac{1}{n-K}\sum_{j=K+1}^{n} X_j X_{j-|k|}$, we have

$$\begin{aligned}
&(2\pi(n-K))^2 \mathrm{Var}[\bar{f}_K(\omega)] \\
&= (n-K)^2 \sum_{l=-K}^{K}\sum_{k=-K}^{K} a_l a_k \exp\{-\mathtt{i}\omega(l-k)\}\mathrm{Cov}(\bar{\sigma}_l, \bar{\sigma}_k) \\
&= \sum_{l=-K}^{K}\sum_{k=-K}^{K} a_l a_k \exp\{-\mathtt{i}\omega(l-k)\} \sum_{r=K+1}^{n}\sum_{s=K+1}^{n} \mathrm{Cov}(X_r X_{r-|l|}, X_s X_{s-|k|}). \qquad (26)
\end{aligned}$$

Now,

$$\begin{aligned}
&\mathrm{Cov}(X_r X_{r-|l|}, X_s X_{s-|k|}) \\
&= \mathbb{E}[X_r X_{r-|l|}]\mathbb{E}[X_s X_{s-|k|}] + \mathbb{E}[X_r X_s]\mathbb{E}[X_{s-|k|} X_{r-|l|}] + \mathbb{E}[X_r X_{s-|k|}]\mathbb{E}[X_s X_{r-|l|}] - \sigma_l \sigma_k \\
&= \sigma_{|r-s|}\sigma_{|r-s-|k|+|l||} + \sigma_{|r-s+|k||}\sigma_{|r-s-|l||}.
\end{aligned}$$

Substituting this into (26), we get

$$\begin{aligned}
&(2\pi(n-K))^2 \mathrm{Var}[\bar{f}_K(\omega)] \\
&= \sum_{l=-K}^{K}\sum_{k=-K}^{K} a_l a_k \exp\{-\mathtt{i}\omega(l-k)\} \sum_{h=-(n-K-1)}^{n-K-1} (n-K-|h|)(\sigma_{|h|}\sigma_{|h-|k|+|l||} + \sigma_{|h+|k||}\sigma_{|h-|l||}) \\
&= \sum_{l=-K}^{K} a_l \exp(-\mathtt{i}\omega l) \sum_{h=-(n-K-1)}^{n-K-1} (n-K-|h|) \sum_{k=-K}^{K} a_k(\sigma_{|h|}\sigma_{|h-k+|l||} + \sigma_{|h+|k||}\sigma_{|h-|l||}) \exp(\mathtt{i}\omega k) \\
&\leq \sum_{l=-K}^{K} a_l \sum_{h=-(n-K-1)}^{n-K-1} (n-K-|h|)(|\sigma_{|h|}| + |\sigma_{|h-|l||}|)M.
\end{aligned}$$

Since $\sum_{h=-(n-K-1)}^{n-K-1}\left(1-\frac{|h|}{n-K}\right)(|\sigma_{|h|}| + |\sigma_{|h-|l||}|) \leq 2M$, it follows $\mathrm{Var}[\bar{f}_K(\omega)] \leq CK(n-K)^{-1}$. For the last term in Equation (25) we have by independence of the ξ_i that

$$\mathbb{E}[C^2] = \frac{1}{4\pi^2(n-K)^2}\sum_{j=K+1}^{n}\mathbb{E}(\tilde{\xi}_j^2) = \frac{8\tilde{\tau}_n}{\alpha^2\pi^2(n-K)}.$$

It remains to calculate $\mathbb{E}[B^2]$. For $j = K+1, ..., n$, we have

$$\begin{aligned}
&\sum_{1\leq|k|\leq K} a_k X_j \xi_{j-|k|}\exp(-\mathtt{i}\omega k) = \sum_{k=1}^{K} a_k X_j \xi_{j-k}\exp(-\mathtt{i}\omega k) + \sum_{k=-K}^{-1} a_k X_j \xi_{j+k}\exp(-\mathtt{i}\omega k) \\
&= \sum_{j-k=j-K}^{j-1} a_k X_j \xi_{j-k}\exp(-\mathtt{i}\omega k) + \sum_{j+k=j-K}^{j-1} a_k X_j \xi_{j+k}\exp(-\mathtt{i}\omega k) \\
&= \sum_{s=j-K}^{j-1} a_{-s+j} X_j \xi_s \exp\{-\mathtt{i}\omega(-s+j)\} + \sum_{t=j-K}^{j-1} a_{t-j} X_j \xi_t \exp\{-\mathtt{i}\omega(t-j)\}.
\end{aligned}$$

Thus,

$$\begin{aligned}2\pi(n-K)B &= \sum_{j=K+1}^{n}\sum_{1\le|k|\le K} a_k X_j \xi_{j-|k|}\exp(-\mathtt{i}\omega k)\\ &= \sum_{j=K+1}^{n} X_j\exp\{-\mathtt{i}\omega j\}\sum_{s=j-K}^{j-1} a_{-s+j}\xi_s\exp\{\mathtt{i}\omega s\}\\ &\quad+\sum_{j=K+1}^{n} X_j\exp\{\mathtt{i}\omega j\}\sum_{t=j-K}^{j-1} a_{t-j}\xi_t\exp\{-\mathtt{i}\omega t\} =: B_1+B_2.\end{aligned}$$

Furthermore,

$$|B_1|^2 = \sum_{j=K+1}^{n}\sum_{h=K+1}^{n} X_jX_h\exp\{-\mathtt{i}\omega(j-h)\}\sum_{s=j-K}^{j-1}\sum_{k=h-K}^{h-1} a_{-s+j}a_{-k+h}\xi_s\xi_k\exp\{-\mathtt{i}\omega(k-s)\},$$

$$|B_2|^2 = \sum_{j=K+1}^{n}\sum_{h=K+1}^{n} X_jX_h\exp\{-\mathtt{i}\omega(h-j)\}\sum_{t=j-K}^{j-1}\sum_{k=h-K}^{h-1} a_{t-j}a_{k-h}\xi_t\xi_k\exp\{-\mathtt{i}\omega(t-k)\}.$$

Using $4\pi^2(n-K)^2|B|^2\le 2(|B_1|^2+|B_2|^2)$, we obtain

$$\begin{aligned}&2\pi^2(n-K)^2\mathbb{E}|B|^2\\ &=\sum_{j=K+1}^{n}\sum_{h=K+1}^{n}\mathbb{E}[X_jX_h]\exp\{-\mathtt{i}\omega(j-h)\}\sum_{s=j-K}^{j-1}\sum_{k=h-K}^{h-1} a_{-s+j}a_{-k+h}\mathbb{E}[\xi_s\xi_k]\exp\{-\mathtt{i}\omega(k-s)\}\\ &\quad+\sum_{j=K+1}^{n}\sum_{h=K+1}^{n}\mathbb{E}[X_jX_h]\exp\{-\mathtt{i}\omega(h-j)\}\sum_{t=j-K}^{j-1}\sum_{k=h-K}^{h-1} a_{t-j}a_{k-h}\mathbb{E}[\xi_t\xi_k]\exp\{-\mathtt{i}\omega(t-k)\}\\ &=\sum_{j=K+1}^{n}\sum_{h=K+1}^{n}\sigma_{|j-h|}\exp\{-\mathtt{i}\omega(j-h)\}\sum_{\substack{s=j-K,\dots,j-1\\ k=h-K,\dots,h-1\\ s=k}} a_{-s+j}a_{-k+h}\frac{32\tau_n^2}{\alpha^2}\\ &\quad+\sum_{j=K+1}^{n}\sum_{h=K+1}^{n}\sigma_{|j-h|}\exp\{-\mathtt{i}\omega(j-h)\}\sum_{\substack{t=j-K,\dots,j-1\\ k=h-K,\dots,h-1\\ t=k}} a_{t-j}a_{k-h}\frac{32\tau_n^2}{\alpha^2}\\ &\le 2\sum_{j=K+1}^{n}\sum_{h=K+1}^{n}\left|\sigma_{|j-h|}\right|\max\{0,K-|j-h|\}\frac{32\tau_n^2}{\alpha^2}\\ &\le\frac{64\tau_n^2(K+1)}{\alpha^2}\sum_{j=K+1}^{n}\sum_{h=K+1}^{n}\left|\sigma_{|j-h|}\right| = \mathcal{O}\left\{32\tau_n^2K(n-K)\alpha^{-2}\right\}\end{aligned}$$

as $|a_{-s+j}a_{-k+h}|\le 1$ and $\sum_{\substack{s=j-K,\dots,j-1\\ k=h-K,\dots,h-1\\ s=k}}\frac{32\tau_n^2}{\alpha^2}=\max\{0,K-|j-h|\}\frac{32\tau_n^2}{\alpha^2}$. Furthermore, we used that $\sum_k|\sigma_k|<M$, see (2). Summing up, we get

$$\mathbb{E}[\{\tilde f_K(\omega)-f(\omega)]\}^2] = \mathcal{O}\left(\frac{K}{n}\right)+\mathcal{O}\left\{\frac{\max(\tau_n^2K,\tilde\tau_n^2)}{(n-K)\alpha^2}\right\}+\mathbb{E}[\mathbb{I}_{A^c}\{\hat f_K(\omega)-f(\omega)\}^2]. \tag{27}$$

To conclude the proof, we show that $\mathbb{E}[\mathbb{I}_{\tilde{A}^c}\{\hat{f}_K(\omega) - f(\omega)\}^2]$ is not larger than the other terms.
L_2 risk on the event $\tilde{A}^c$. Application of the Cauchy-Schwarz inequality gives

$$\mathbb{E}[\mathbb{I}_{\tilde{A}^c}\{\hat{f}_K(\omega) - f(\omega)\}^2] \leq \left(\mathbb{E}[\{\hat{f}_K(\omega) - f(\omega)\}^4] \cdot \Pr[\tilde{A}^c]\right)^{1/2}.$$

Using $\|f\|_\infty < M$ and $|\widetilde{V}_j| \leq \tilde{\tau}_n$ yields for some constants $c_1, ..., c_5 > 0$

$$\begin{aligned}
\mathbb{E}[\{\hat{f}_K(\omega) - f(\omega)\}^4] &\leq c_1 \mathbb{E}[\hat{f}_K(\omega)^4] + c_2 \\
&\leq \frac{c_3}{\{2\pi(n-K)\}^4} \left\{ \mathbb{E}\left[\left(\sum_{j=K+1}^n \widetilde{V}_j\right)^4\right] + \mathbb{E}\left[\left(\sum_{j=K+1}^n \tilde{\xi}_j\right)^4\right] \right\} + c_2 \\
&\leq c_4 \tilde{\tau}_n^4 + \frac{c_5 \tilde{\tau}_n^4}{\alpha^4} + c_2 = \mathcal{O}(\tilde{\tau}_n^4 + \tilde{\tau}_n^4 \alpha^{-4}).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\Pr[\tilde{A}^c] &\leq \sum_{i=1}^n \Pr[\{\widetilde{X}_i \neq X_i \text{ or } (\widetilde{V}_i \neq V_i) \cdot \mathbf{1}(i > K)\}] \\
&\leq \sum_{i=1}^n \Pr[\widetilde{X}_i \neq X_i] + \Pr[\{\widetilde{X}_i = X_i \text{ and } (\widetilde{V}_i \neq V_i) \cdot I(i > K)\}].
\end{aligned}$$

From (24), we know that $\sum_{i=1}^n \Pr[\widetilde{X}_i \neq X_i] = \mathcal{O}(n^{-3})$ for $\tau_n^2 \geq 8 \log^{1+\delta}(n)$. Next, we need to bound from above, for $i > K$:

$$\begin{aligned}
&\Pr[\{\widetilde{X}_i = X_i \text{ and } \widetilde{V}_i \neq V_i\}] \\
&\leq \Pr\left[\left|\widetilde{X}_i^2 + \widetilde{X}_i \sum_{1 \leq |k| \leq K} a_k (\widetilde{X}_{i-|k|} + \xi_{i-|k|}) \exp(-\mathtt{i}\omega k)\right| \geq \tilde{\tau}_n\right] \\
&= \Pr\left[|\widetilde{X}_i| \cdot \left|\sum_{0 \leq |k| \leq K} a_k (\widetilde{X}_{i-|k|} + \xi_{i-|k|}) \exp(-\mathtt{i}\omega k)\right| \geq \tilde{\tau}_n\right] \\
&\lesssim \Pr\left[\left|\sum_{0 \leq |k| \leq K} a_k \widetilde{X}_{i-|k|} \exp(-\mathtt{i}\omega k) + \sum_{0 \leq |k| \leq K} a_k \xi_{i-|k|} \exp(-\mathtt{i}\omega k)\right| \geq \frac{\tilde{\tau}_n}{\tau_n}\right] \\
&\lesssim \Pr\left[\left|\sum_{0 \leq |k| \leq K} a_k \widetilde{X}_{i-|k|} \cos(\omega k)\right| \geq \frac{1}{2}\frac{\tilde{\tau}_n}{\tau_n}\right] + \Pr\left[\left|\sum_{0 \leq |k| \leq K} a_k \xi_{i-|k|} \cos(\omega k)\right| \geq \frac{1}{2}\frac{\tilde{\tau}_n}{\tau_n}\right] \\
&\lesssim \Pr\left[\left|\sum_{0 \leq k \leq K} a_k \widetilde{X}_{i-k} \cos(\omega k)\right| \geq \frac{1}{4}\frac{\tilde{\tau}_n}{\tau_n}\right] + \Pr\left[\left|\sum_{0 \leq k \leq K} a_k \xi_{i-k} \cos(\omega k)\right| \geq \frac{1}{4}\frac{\tilde{\tau}_n}{\tau_n}\right] \\
&\lesssim \Pr\left[\left|\sum_{0 \leq k \leq K} X_{i-k}\right| \geq \frac{1}{4}\frac{\tilde{\tau}_n}{\tau_n}\right] + \Pr\left[\left|\sum_{0 \leq k \leq K} \xi_{i-k}\right| \geq \frac{1}{4}\frac{\tilde{\tau}_n}{\tau_n}\right]. \qquad (28)
\end{aligned}$$

In particular, the first sum consists of mean zero, Gaussian random variables and the second sum of i.i.d. mean zero Laplace random variables. For the variance of the Gaussian sum we obtain

$$\operatorname{Var}\left(\sum_{0\le k\le K} X_{i-k}\right) = \sum_{0\le k\le K}\sum_{0\le l\le K} \operatorname{Cov}\left(X_{i-k}, X_{i-l}\right) = \sum_{0\le k\le K}\sum_{0\le l\le K} \sigma_{|k-l|} \le M(K+1).$$

This implies

$$\Pr\left[\left|\sum_{0\le k\le K} X_{i-k}\right| \ge \frac{1}{4}\frac{\tilde\tau_n}{\tau_n}\right] \le 2\exp\left(-\frac{\tilde\tau_n^2}{32M\tau_n^2(K+1)}\right),$$

which is of the order $o(n^{-3})$ for $\tilde\tau_n^2\tau_n^{-2} \ge 96(K+1)\log^{1+\delta}(n)$. To bound the second term in (28) we apply the following version of the Bernstein inequality: Let $Y_1,...,Y_n$ independent zero-mean random variables. If for some $L>0$ and every integer $k\ge 2$, $\mathbb{E}\left[\left|Y_i^k\right|\right] \le \frac{1}{2}\mathbb{E}\left[Y_i^2\right]L^{k-2}k!$, then

$$\Pr\left(\sum_{i=1}^n Y_i \ge 2t\sqrt{\sum_{i=1}^n \mathbb{E}\left[Y_i^2\right]}\right) < \exp(-t^2), \qquad \text{for} \quad 0\le t\le \frac{1}{2L}\sqrt{\sum_{i=1}^n \mathbb{E}\left[Y_i^2\right]}.$$

In our case $\xi_0,...,\xi_K \overset{\text{i.i.d.}}{\sim} \text{Lap}(b)$ with $b = 4\tau_n/\alpha$. Recall that $\mathbb{E}[|\xi_i^k|] \le \mathbb{E}[\xi_i^2]k!b^{k-2}/2$ and thus $L=b$. Then, by symmetry

$$\Pr\left[\left|\sum_{0\le k\le K}\xi_k\right| \ge 8\sqrt{2}t\sqrt{K+1}\tau_n/\alpha\right] \le 2\exp(-t^2), \qquad \text{for} \quad 0\le t\le\sqrt{(K+1)/2}.$$

If $\tilde\tau_n$ and τ_n are chosen such that

$$0 \le \frac{\alpha\tilde\tau_n}{32\sqrt{2}\sqrt{K+1}\tau_n^2} \le \sqrt{(K+1)/2}, \tag{29}$$

then

$$\Pr\left[\left|\sum_{0\le k\le K}\xi_k\right| \ge \frac{1}{4}\frac{\tilde\tau_n}{\tau_n}\right] \le 2\exp\left(-\frac{\tilde\tau_n^2\alpha^2}{2048(K+1)\tau_n^4}\right). \tag{30}$$

Set $\tau_n^2 = 8\log^{1+\delta}(n)$ for some $\delta>0$ and $\tilde\tau_n^2 = 1024\tau_n^6(K+1)$. In particular, $\tilde\tau_n^2\tau_n^{-2} \ge 96(K+1)\log^{1+\delta}(n)$. If K is chosen such that

$$\alpha\tau_n \le \sqrt{(K+1)}, \tag{31}$$

then condition (29) is satisfied for this choice of τ_n^2 and $\tilde\tau_n^2$. Hence, the right-hand side of (30) is of order $\mathcal{O}\left\{\exp\left(-4\log(n)^{1+\delta}\alpha^2\right)\right\} = \mathcal{O}(n^{-4})$, as for every $\alpha,\delta>0$ there exists an $n_{\alpha,\delta}$ such that for all $n\ge n_{\alpha,\delta}$ it holds that $\log(n)^\delta\alpha^2\ge 1$. Thus, we have shown that for sufficiently large n,

$$\mathbb{E}[\mathbb{I}_{\tilde A^c}\{\hat f_K(\omega) - f(\omega)\}^2] \le \frac{c}{n}. \tag{32}$$

Combining (27) and (32), setting $\tau_n^2 = 8\log^{1+\delta}(n)$ and $\tilde{\tau}_n^2 = 1024\tau_n^6(K+1)$, and choosing K such that condition (31) is fulfilled gives

$$\mathbb{E}|\hat{f}_K(\omega) - f(\omega)|^2 \quad = \mathcal{O}\left(\frac{K}{n}\right) + \mathcal{O}\left(\frac{K\tau_n^6}{(n-K)\alpha^2}\right) + \begin{cases}\mathcal{O}(K^{-2s}), & f \in W^{s,\infty}(L_0, L),\\ \mathcal{O}(K^{-2s+1}), & f \in W^{s,2}(L).\end{cases}$$

Optimizing $K^{-2s} + K \cdot (1 \vee \tau_n^6\alpha^{-2})/n$ with respect to K gives

$$K = \left(\frac{1}{n} \vee \frac{\tau_n^6}{n\alpha^2}\right)^{-\frac{1}{2s+1}}.$$

Analogously, optimizing $K^{-2s+1} + K \cdot (1 \vee \tau_n^6\alpha^{-2})/n$ with respect to K gives

$$K = \left(\frac{1}{n} \vee \frac{\tau_n^6}{n\alpha^2}\right)^{-\frac{1}{2s}}.$$

As both choices of K satisfy condition (31) for sufficiently large n, and $f \in W^{s,\infty}(L_0, L)$ resp. $f \in W^{s,2}(L)$ was chosen arbitrary, we obtain

$$\sup_{f \in W^{s,\infty}(L_0,L)} \mathbb{E}|\hat{f}_K(\omega) - f(\omega)|^2 \leq c_1 \left[\frac{1}{n} \vee \frac{\log^{3+3\delta}(n)}{n\alpha^2}\right]^{\frac{2s}{2s+1}}$$

and

$$\sup_{f \in W^{s,2}(L)} \mathbb{E}|\hat{f}_K(\omega) - f(\omega)|^2 \leq c_2 \left[\frac{1}{n} \vee \frac{\log^{3+3\delta}(n)}{n\alpha^2}\right]^{\frac{2s-1}{2s}}.$$

■

A.6 Proof of Lemma 6

Proof of i): Define $\check{Z}_i = 0$ for $i = 1, .., K$ and denote $\tilde{x}$ for x trimmed at $\pm\tau_n$. Then, for $i = 1, ..., n$, for all z and $\check{z}$ and for all x different from x', it holds

$$\begin{aligned}
&\frac{q^{(Z_i,\check{Z}_i)|X_i=x,(Z_1,\check{Z}_1),\dots,(Z_{i-1},\check{Z}_{i-1})}(z,\check{z})}{q^{(Z_i,\check{Z}_i)|X_i=x',(Z_1,\check{Z}_1),\dots,(Z_{i-1},\check{Z}_{i-1})}(z,\check{z})}\\
&= \frac{\frac{\alpha}{8\tau_n}\exp\left(-\frac{\alpha}{4\tau_n}|z-\tilde{x}|\right) q^{\check{Z}_i|X_i=x,Z_{1:(i-1)},\check{Z}_{1:(i-1)}}(\check{z})}{\frac{\alpha}{8\tau_n}\exp\left(-\frac{\alpha}{4\tau_n}|z-\tilde{x}'|\right) q^{\check{Z}_i|X_i=x',Z_{1:(i-1)},\check{Z}_{1:(i-1)}}(\check{z})}\\
&= \frac{\exp\left(-\frac{\alpha}{4\tau_n}|z-\tilde{x}|\right) q^{\check{Z}_i|X_i=x,W_i}(\check{z})}{\exp\left(-\frac{\alpha}{4\tau_n}|z-\tilde{x}'|\right) q^{\check{Z}_i|X_i=x',W_i}(\check{z})} \leq e^{\alpha/2}\frac{q^{\check{Z}_i|X_i=x,W_i}(\check{z})}{q^{\check{Z}_i|X_i=x',W_i}(\check{z})},
\end{aligned}$$

where we used $|\tilde{x} - \tilde{x}'| \leq 2\tau_n$. The proof that $\check{Z}_i$ is an $\alpha/2$-LDP view of X_i, W_i, i.e., that for all $\check{z}$ and all $x \neq x' \in \mathbb{R}^d$ it holds

$$\frac{q^{\check{Z}_i|X_i=x,W_i}(\check{z})}{q^{\check{Z}_i|X_i=x',W_i}(\check{z})} = \frac{q^{\check{Z}_i|X_i=x,\widetilde{W}_i}(\check{z})}{q^{\check{Z}_i|X_i=x',\widetilde{W}_i}(\check{z})} \leq e^{\alpha/2},$$

follows from results of Duchi et al. (2018) and Butucea et al. (2023).

Proof of ii): By construction, for $i = K+2, ..., n$, conditioned on $X_i, Z_{1:(i-1)}$, the $\check{Z}_i$ is independent from $\check{Z}_l$, $l = K+1, ..., i-1$. Thus,

$$\mathbb{E}[\check{Z}_i \mid X_i, Z_{1:(i-1)}, \check{Z}_{K+1:(i-1)}] = \mathbb{E}[\check{Z}_i \mid X_i, Z_{1:(i-1)}].$$

Next,

$$\mathbb{E}[\check{Z}_i \mid X_i, Z_{1:(i-1)}] = \mathbb{E}[\check{Z}_i \mid \widetilde{W}_i] = \mathbb{E}[\mathbb{E}[\check{Z}_i \mid \widetilde{Y}_i] \mid \widetilde{W}_i].$$

By the same arguments as in the proof of Proposition 3.2. of Butucea et al. (2023), one can show that almost surely

$$\mathbb{E}[\check{Z}_i \mid \widetilde{Y}_i] = \widetilde{Y}_i.$$

Furthermore, for every $j = 0, .., K$,

$$\mathbb{E}[\widetilde{Y}_{i,j} \mid \widetilde{W}_{i,j} = w_j] = \tilde{\tau}_n \left(\frac{1}{2} + \frac{w_j}{2\tilde{\tau}_n}\right) - \tilde{\tau}_n \left(\frac{1}{2} - \frac{w_j}{2\tilde{\tau}_n}\right) = w_j.$$

Hence, the mechanism is conditionally unbiased. Next,

$$\begin{aligned}
&\mathrm{Var}(\check{Z}_{i,j} \mid X_i, Z_{1:(i-1)}, \check{Z}_{1:(i-1)}) \\
&= \mathbb{E}\left[(\check{Z}_{i,j} - \mathbb{E}[\check{Z}_{i,j} \mid X_i, Z_{1:(i-1)}, \check{Z}_{1:(i-1)}])^2 \mid X_i, Z_{1:(i-1)}, \check{Z}_{1:(i-1)}\right] \\
&= \mathbb{E}\left[(\check{Z}_{i,j} - \mathbb{E}[\check{Z}_{i,j} \mid X_i, Z_{1:(i-1)}])^2 \mid X_i, Z_{1:(i-1)}\right] = \mathrm{Var}(\check{Z}_{i,j} \mid X_i, Z_{1:(i-1)}) \leq \mathrm{Var}(\check{Z}_{i,j}) \leq B^2,
\end{aligned}$$

since $\check{Z}_{i,j} \in \{-B, B\}$ for all $i = K+1, ..., n$ and $j = 0, ..., K$. Finally, it is shown by Butucea et al. (2023, Lemma B.1.) that C_K behaves asymptotically as $\sqrt{\pi/2}\sqrt{K}$, which implies $B = \mathcal{O}(\tilde{\tau}_n \sqrt{K} \alpha^{-1})$ for $K \to \infty$. ∎

A.7 Proof of Theorem 7

We rewrite the estimator $\check{f}_K$ as

$$\begin{aligned}
\check{f}_K(\omega) &= \frac{1}{2\pi(n-K)} \sum_{i=K+1}^{n} \sum_{0 \leq |k| \leq K} \check{Z}_{i,|k|} \exp(-\mathtt{i}k\omega) \\
&= \frac{1}{2\pi(n-K)} \sum_{0 \leq |k| \leq K} \sum_{i=K+1}^{n} \check{Z}_{i,|k|} \exp(-\mathtt{i}k\omega) \\
&= \frac{1}{2\pi} \sum_{0 \leq |k| \leq K} \check{\sigma}_k \exp(-\mathtt{i}k\omega), \quad \omega \in [-\pi, \pi],
\end{aligned} \tag{33}$$

where $\check{Z}_{i,|k|}$ is the $|k|$-th entry of $\check{Z}_i$ and $\check{\sigma}_j = \frac{1}{n-K} \sum_{i=K+1}^{n} \check{Z}_{i,|j|}$.
Let $f_K(\omega) = (2\pi)^{-1} \sum_{0 \leq |k| \leq K} \sigma_k \exp(-\mathtt{i}k\omega)$. Then, by Parseval's identity, (17) and (18),

$$\begin{aligned}
\mathbb{E}\|\check{f}_K - f\|_2^2 &= \mathbb{E}\|\check{f}_K - f_K\|_2^2 + \|f_K - f\|_2^2 \\
&= \sum_{0 \leq |j| \leq K} \mathbb{E}[(\check{\sigma}_j - \sigma_j)^2] + \begin{cases} \mathcal{O}(K^{-2s}), & f \in W^{s,\infty}(L_0, L), \\ \mathcal{O}(K^{-2s}), & f \in W^{s,2}(L). \end{cases}
\end{aligned}$$

It is sufficient to consider $j \in \{0, 1, ..., K\}$. In particular,

$$\mathbb{E}[(\check{\sigma}_j - \sigma_j)^2] = \mathbb{E}[(\check{\sigma}_j - \mathbb{E}[\check{\sigma}_j])^2] + (\mathbb{E}[\check{\sigma}_j] - \sigma_j)^2 = \mathrm{Var}(\check{\sigma}_j) + (\mathbb{E}[\check{\sigma}_j] - \sigma_j)^2.$$

By the law of total variance,

$$\mathrm{Var}(\check{\sigma}_j) = \mathbb{E}[\mathrm{Var}(\check{\sigma}_j \mid \tilde{W}_{(K+1):n})] + \mathrm{Var}(\mathbb{E}[\check{\sigma}_j \mid \tilde{W}_{(K+1):n}]).$$

Since conditioned on $\widetilde{W}_{K+1}, .., \tilde{W}_n$, the $\check{Z}_{i,|j|}$, $i = K+1, ..., n$, are independent, we have

$$\begin{aligned}\mathbb{E}[\mathrm{Var}(\check{\sigma}_j \mid \tilde{W}_{(K+1):n})] &= \mathbb{E}\left[\frac{1}{(n-K)^2} \sum_{i=K+1}^{n} \mathrm{Var}(\check{Z}_{i,j} \mid \tilde{W}_{(K+1):n})\right] \\ &\leq \frac{1}{(n-K)^2} \sum_{i=K+1}^{n} \mathrm{Var}(\check{Z}_{i,j}) \leq \frac{B^2}{n-K},\end{aligned}$$

by Lemma 6. Again, with Lemma 6 it follows, that

$$\mathbb{E}[\check{Z}_i \mid \widetilde{W}_{(K+1):n}] = \mathbb{E}[\check{Z}_i \mid \widetilde{W}_{(K+1):i}] = \mathbb{E}[\check{Z}_i \mid X_i, Z_{1:(i-1)}] = \widetilde{W}_i,$$

and hence

$$\mathbb{E}[\check{\sigma}_j \mid \widetilde{W}_{(K+1):n}] = \frac{1}{n-K} \sum_{i=K+1}^{n} \mathbb{E}[\check{Z}_{i,j} \mid \widetilde{W}_{(K+1):n}] = \frac{1}{n-K} \sum_{i=K+1}^{n} \widetilde{W}_{i,j}. \tag{34}$$

Similar calculations as in the proof of Theorem 3 yield that

$$\mathrm{Var}\left(\frac{1}{n-K} \sum_{i=K+1}^{n} \widetilde{W}_{i,j}\right) \leq \frac{2M_1}{(n-K)} + \frac{8M\tau_n^2}{\alpha^2(n-K)} + \frac{c\tilde{\tau}_n^2}{n-K},$$

if $\tau_n^2 = 8\log^{1+\delta}(n)$ and $\widetilde{\tau}_n \geq 16\log^{1+\delta}(n)\tau_n^2$ for some $\delta > 0$ is chosen. Thus,

$$\mathrm{Var}(\check{\sigma}_j) \leq \frac{2M_1}{(n-K)} + \frac{8M\tau_n^2}{\alpha^2(n-K)} + \frac{c\tilde{\tau}_n^2}{n^2} + \frac{B^2}{n-K}.$$

To compute the bias $(\mathbb{E}[\check{\sigma}_j] - \sigma_j)$, we define, analogous to the proof of Theorem 3, the event A_j and its complement:

$$A_j = \bigcap_{i=1}^{n} \{\widetilde{X}_i = X_i \text{ and } (\widetilde{W}_{i,j} = W_{i,j}) I(i > K)\}, \quad A_j^c = \bigcup_{i=1}^{n} \{\widetilde{X}_i \neq X_i \text{ or } (\widetilde{W}_{i,j} \neq W_{i,j}) I(i > K)\}.$$

Using (34), we get

$$\begin{aligned}(\mathbb{E}[\check{\sigma}_j] - \sigma_j)^2 &= \left(\mathbb{E}\left[\frac{1}{n-K} \sum_{i=K+1}^{n} \widetilde{W}_{i,j} - \sigma_j\right]\right)^2 \\ &\lesssim \mathbb{E}\left[\mathbb{I}_{A_j}\left(\frac{1}{n-K} \sum_{i=K+1}^{n} \widetilde{W}_{i,j} - \sigma_j\right)^2\right] + \mathbb{E}\left[\mathbb{I}_{A_j^c}\left(\frac{1}{n-K} \sum_{i=K+1}^{n} \widetilde{W}_{i,j} - \sigma_j\right)^2\right].\end{aligned}$$

In particular, if the event A_j occurs, then

$$\mathbb{E}\left[\mathbb{I}_{A_j}\left(\frac{1}{n-K}\sum_{i=K+1}^{n}\widetilde{W}_{i,j}-\sigma_j\right)^2\right]=\mathbb{E}[\mathbb{I}_{A_j}(\tilde{\sigma}_j-\sigma_j)^2],$$

where $\tilde{\sigma}_j$ is the untruncated estimator defined by

$$\tilde{\sigma}_j=\frac{1}{n-K}\sum_{i=K+1}^{n}X_i\cdot(X_{i-j}+\xi_{i-j})\,.$$

Thus,

$$(\mathbb{E}[\check{\sigma}_j]-\sigma_j)^2\lesssim\mathbb{E}[\mathbb{I}_{A_j}(\tilde{\sigma}_j-\sigma_j)^2]+\left(\Pr(A_j^c)\mathbb{E}\left[\left(\frac{1}{n-K}\sum_{i=K+1}^{n}\widetilde{W}_{i,j}-\sigma_j\right)^4\right]\right)^{1/2}.$$

Similar calculations as in the proof of Theorem 3 yield

$$\mathbb{E}[\mathbb{I}_{A_j}(\tilde{\sigma}_j-\sigma_j)^2]\leq\frac{2M_1}{(n-K)}+\frac{8M\tau_n^2}{\alpha^2(n-K)}$$

and that $\Pr(A_j^c)=\mathcal{O}(n^{-3})$, if $\tau_n^2=8\log^{1+\delta}(n)$ and $\tilde{\tau}_n=16\log^{1+\delta}(n)\tau_n^2$ for some $\delta>0$ is chosen. Since $|\widetilde{W}_{i,j}|\leq\tilde{\tau}_n$, it follows that

$$\mathbb{E}[(\check{\sigma}_j-\sigma_j)^2]\leq\frac{2M_1}{n-K}+\frac{8M\tau_n^2}{\alpha^2(n-K)}+\frac{cB^2}{n-K}+o(n^{-1}).$$

By Stirling's approximation, $B\lesssim\tilde{\tau}_n\sqrt{K+1}/\alpha$ and thus

$$\mathbb{E}\|\check{f}_K-f\|_2^2\leq C\left\{\frac{K}{n-K}+\frac{K\tau_n^2}{\alpha^2(n-K)}+\frac{\tilde{\tau}_n^2K^2}{\alpha^2(n-K)}+\frac{1}{K^{2s}}\right\}.$$

Optimizing $K^{-2s}+K^2\tilde{\tau}_n^2/(\alpha^2 n)+K/n$ with respect to K gives $K\asymp\{\tilde{\tau}_n^2/(n\alpha^2)\}^{-1/(2s+2)}\wedge n^{-1/(2s+1)}$. As $f\in W^{s,\infty}(L_0,L)$ resp. $f\in W^{s,2}(L)$ was chosen arbitrary, and $\alpha>0$, we obtain

$$\sup_{f\in W^{s,\infty}(L_0,L)}\mathbb{E}\|\check{f}_K-f\|^2\leq c_1\left\{\left[\frac{\log^{4+4\delta}(n)}{n\alpha^2}\right]^{\frac{2s}{2s+2}}\vee\left(\frac{1}{n}\right)^{\frac{2s}{2s+1}}\right\},$$

$$\sup_{f\in W^{s,2}(L)}\mathbb{E}\|\check{f}_K-f\|^2\leq c_2\left\{\left[\frac{\log^{4+4\delta}(n)}{n\alpha^2}\right]^{\frac{2s}{2s+2}}\vee\left(\frac{1}{n}\right)^{\frac{2s}{2s+1}}\right\}.$$

∎

A.8 Proof of Corollary 8

Let $\omega \in [-\pi, \pi]$. Then,

$$\mathbb{E}|\check{f}_K(\omega) - f(\omega)|^2 = \{\mathbb{E}[\check{f}_K(\omega)] - f(\omega)\}^2 + \mathbb{E}(\check{f}_K(\omega) - \mathbb{E}[\check{f}_K(\omega)])^2.$$

Using (33) and (34) yields

$$\begin{aligned}
\{\mathbb{E}[\check{f}_K(\omega)] - f(\omega)\}^2 &= \left\{ \frac{1}{2\pi} \sum_{0 \le |j| \le K} \mathbb{E}[\check{\sigma}_j] \exp(-\mathtt{i} j\omega) - f(\omega) \right\}^2 \\
&= \left\{ \frac{1}{2\pi} \sum_{0 \le |j| \le K} \mathbb{E}[\mathbb{E}[\check{\sigma}_j \mid \widetilde{W}_{(K+1):n}]] \exp(-\mathtt{i} j\omega) - f(\omega) \right\}^2 \\
&= \left\{ \mathbb{E}\left[\frac{1}{2\pi(n-K)} \sum_{0 \le |j| \le K} \sum_{i=K+1}^{n} \widetilde{W}_{i,|j|} \exp(-\mathtt{i} j\omega) \right] - f(\omega) \right\}^2 .
\end{aligned}$$

In particular, if the event $\cup_{j=0}^{K} A_j$ without any truncation occurs, then the estimator

$$\frac{1}{2\pi(n-K)} \sum_{0 \le |j| \le K} \sum_{i=K+1}^{n} \widetilde{W}_{i,|j|} \exp(-\mathtt{i} j\omega) \tag{35}$$

equals the estimator $\hat{f}_K^{\mathrm{NI}}(\omega)$ in Section 2.1, but with less Laplace noise. The probability of the event that truncation occurs can be computed similarly as in the proof of Theorem 3. Combining these results, we obtain the following bound for the bias: if $\tau_n^2 = 8\log^{1+\delta}(n)$ and $\tilde{\tau}_n = 16\log^{1+\delta}(n)\tau_n^2$ for some $\delta > 0$, then

$$\{\mathbb{E}[\check{f}_K(\omega)] - f(\omega)\}^2 \lesssim \frac{K}{n} + \begin{cases} \log(K) K^{-2s}, & f \in W^{s,\infty}(L_0, L), \\ K^{-2s+1}, & f \in W^{s,2}(L), \end{cases}$$

where the constant $C > 0$ depends on s, L, if $f \in W^{s,2}(L)$, and additionally on L_0, if $f \in W^{s,\infty}(L_0, L)$. Next, by the law of total variance,

$$\begin{aligned}
\mathrm{Var}\left(\check{f}_K(\omega)\right) &= \mathrm{Var}\left(\frac{1}{2\pi} \sum_{0 \le |k| \le K} \check{\sigma}_k \exp(-\mathtt{i} k\omega) \right) \\
&= \mathbb{E}\left[\mathrm{Var}\left(\frac{1}{2\pi} \sum_{0 \le |k| \le K} \check{\sigma}_k \exp(-\mathtt{i} k\omega) \mid \widetilde{W}_{(K+1):n} \right) \right] \\
&\quad + \mathrm{Var}\left(\mathbb{E}\left[\frac{1}{2\pi} \sum_{0 \le |k| \le K} \check{\sigma}_k \exp(-\mathtt{i} k\omega) \mid \widetilde{W}_{(K+1):n} \right] \right).
\end{aligned}$$

As $\mathbb{E}[\check{\sigma}_j \mid \widetilde{W}_{(K+1):n}] = \frac{1}{n-K} \sum_{i=K+1}^{n} \widetilde{W}_{i,|j|}$, again, the second variance term above can be bounded by similar calculations as for $\hat{f}_K^{\mathrm{NI}}(\omega)$ in the proof of Proposition 1. In particular, the terms with α^4 do not appear, since (35) includes less added Laplace noise. The probability of the event that truncation occurs can be computed similarly as in the proof of Theorem 3. Thus,

$$\operatorname{Var}\left(\mathbb{E}\left[\frac{1}{2\pi}\sum_{0\le|k|\le K}\check{\sigma}_k\exp(-\mathtt{i}k\omega)\mid\widetilde{W}_{(K+1):n}\right]\right)\lesssim\frac{K}{n}+\frac{\tau_n^2K}{n\alpha^2}.$$

Since conditioned on $W_{(K+1):n}$, the $\check{Z}_{i,|k|}$ for $i=K+1,...,n$ and $k=0,...,K$ are independent, it follows

$$\begin{aligned}&\operatorname{Var}\left(\frac{1}{2\pi}\sum_{0\le|k|\le K}\check{\sigma}_k\exp(-\mathtt{i}k\omega)\mid\widetilde{W}_{(K+1):n}\right)\\&=\frac{1}{2\pi}\sum_{0\le|k|\le K}\frac{1}{(n-K)^2}\sum_{i=K+1}^{n}\operatorname{Var}\left(\check{Z}_{i,|k|}\mid\widetilde{W}_{(K+1):n}\right)\lesssim\frac{KB^2}{2\pi(n-K)},\end{aligned}$$

where we used that $\check{Z}_{i,|k|}\in\{-B,B\}$. Combining the previous bounds results in the following bound for the variance

$$\operatorname{Var}\left(\check{f}_K(\omega)\right)=\mathcal{O}\left(\frac{K}{n}\right)+\mathcal{O}\left(\frac{\tau_n^2K}{n\alpha^2}\right)+\mathcal{O}\left(\frac{KB^2}{2\pi(n-K)}\right).$$

Using $B\lesssim\tilde{\tau}_n\sqrt{K+1}/\alpha$ by Lemma 6, we obtain for $\tau_n^2=8\log^{1+\delta}(n)$ and $\tilde{\tau}_n=16\log^{1+\delta}(n)\tau_n^2$ for some $\delta>0$, that

$$\begin{aligned}\mathbb{E}|\check{f}_K(\omega)-f(\omega)|^2=&\mathcal{O}\left(\frac{K}{n}\right)+\mathcal{O}\left(\frac{K\tau_n^2}{n\alpha^2}\right)+\mathcal{O}\left(\frac{K^2\tilde{\tau}_n^2}{2\pi(n-K)\alpha^2}\right)\\&+\begin{cases}\mathcal{O}(\log(K)K^{-2s}), & f\in W^{s,\infty}(L_0,L),\\\mathcal{O}(K^{-2s+1}), & f\in W^{s,2}(L).\end{cases}\end{aligned}$$

Optimizing with respect to K completes the proof. ■

A.9 Proof of Lemma 9

Now, we prove Equation (9), that is, for all $i=1,\dots,n$,

$$I(Z_i|Z_{1:(i-1)})\lesssim(e^{\alpha}-1)^2I(X_i|Z_{1:(i-1)}),$$

where, by convention, there is no conditioning when $i=1$. Note that

$$\mathcal{L}_\theta^{Z_i|Z_{1:(i-1)}}(z_i)=\int q^{Z_i|X_i,Z_{1:(i-1)}}(z_i)\mathcal{L}_\theta^{X_i|Z_{1:(i-1)}}\,\mathrm{d}x_i\ge\inf_x q^{Z_i|X_i=x,Z_{1:(i-1)}}(z_i). \tag{36}$$

Moreover, $\int\dot{\mathcal{L}}_\theta^{X_i|Z_{1:(i-1)}}(z)\,\mathrm{d}z=0$ implies that

$$\int\left[\dot{\mathcal{L}}_\theta^{X_i|Z_{1:(i-1)}}(x_i)\right]_+\mathrm{d}x_i=\int\left[\dot{\mathcal{L}}_\theta^{X_i|Z_{1:(i-1)}}(x_i)\right]_-\mathrm{d}x_i$$

$$= \frac{1}{2} \int \left| \dot{\mathcal{L}}_\theta^{X_i|Z_{1:(i-1)}}(x_i) \right| \mathrm{d}x_i \leq \frac{1}{2} I^{1/2}(X_i|Z_{1:(i-1)}).$$

Thus, see also Duchi et al.,

$$\begin{aligned}
\dot{\mathcal{L}}_\theta^{Z_i|Z_{1:(i-1)}}(z_i) &= \int q^{Z_i|X_i,Z_{1:(i-1)}}(z_i) \dot{\mathcal{L}}_\theta^{X_i|Z_{1:(i-1)}}(x_i) \,\mathrm{d}x_i \\
&= \int q^{Z_i|X_i,Z_{1:(i-1)}}(z_i) \left(\left[\dot{\mathcal{L}}_\theta^{X_i|Z_{1:(i-1)}}(x_i) \right]_+ - \left[\dot{\mathcal{L}}_\theta^{X_i|Z_{1:(i-1)}}(x_i) \right]_- \right) \mathrm{d}x_i \\
&\leq |\sup_x q^{Z_i|X_i=x,Z_{1:(i-1)}}(z_i) - \inf_x q^{Z_i|X_i=x,Z_{1:(i-1)}}(z_i)| \frac{1}{2} \int \left| \dot{\mathcal{L}}_\theta^{X_i|Z_{1:(i-1)}}(x_i) \right| \mathrm{d}x_i \\
&\leq (e^\alpha - 1) \cdot (2 \wedge e^\alpha) \cdot \inf_x q^{Z_i|X_i=x,Z_{1:(i-1)}}(z_i) \cdot \frac{1}{2} \int \left| \dot{\mathcal{L}}_\theta^{X_i|Z_{1:(i-1)}}(x_i) \right| \mathrm{d}x_i.
\end{aligned} \tag{37}$$

We conclude that (9) holds by bounding $(2 \wedge e^\alpha) \cdot \frac{1}{2} \leq 1$ and by using (36) and (37) that

$$\begin{aligned}
I(Z_i|Z_{1:(i-1)}) &\leq (e^\alpha - 1)^2 \mathbb{E}_{Z_{1:(i-1)}} \left[\int \inf_x q^{Z_i|X_i=x,Z_{1:(i-1)}}(z_i) \,\mathrm{d}z_i \left(\int \left| \dot{\mathcal{L}}_\theta^{X_i|Z_{1:(i-1)}}(x_i) \right| \mathrm{d}x_i \right)^2 \right] \\
&\leq (e^\alpha - 1)^2 I(X_i|Z_{1:(i-1)}).
\end{aligned}$$

For $i \geq 2$, we prove (10), that is, for an arbitrary $\epsilon > 0$,

$$I(X_i|Z_{1:(i-1)}) \leq (1 + \frac{1}{\epsilon})(e^\alpha - 1)^2 I(X_i, X_{i-1}|Z_{1:(i-2)}) + (1 + \epsilon) e^{2\alpha} I(X_i|Z_{1:(i-2)}).$$

Thus,

$$\begin{aligned}
I(X_i|Z_{1:(i-1)}) &= I(X_i, Z_{1:(i-1)}) - I(Z_{1:(i-1)}) = I(X_i, Z_{i-1}|Z_{1:(i-2)}) + I(Z_{1:(i-2)}) - I(Z_{1:(i-1)}) \\
&\leq \mathbb{E}_{Z_{1:(i-2)}} \left[\int \frac{(\dot{\mathcal{L}}_\theta^{X_i,Z_{i-1}|Z_{1:(i-2)}}(x_i, z_{i-1}))^2}{\mathcal{L}_\theta^{X_i,Z_{i-1}|Z_{1:(i-2)}}(x_i, z_{i-1})} \,\mathrm{d}x_i \,\mathrm{d}z_{i-1} \right] - I(Z_{1:(i-1)}|Z_{1:(i-2)}).
\end{aligned}$$

We use now that, conditionally on X_{i-1}, the variables X_i and Z_{i-1} are independent to write that

$$\begin{aligned}
\mathcal{L}_\theta^{X_i,Z_{i-1}|Z_{1:(i-2)}}(x_i, z_{i-1}) &= \int q^{Z_{i-1}|X_{i-1},Z_{1:(i-2)}}(z_{i-1}) \cdot \mathcal{L}_\theta^{X_i,X_{i-1}|Z_{1:(i-2)}}(x_i, x_{i-1}) \,\mathrm{d}x_{i-1} \\
&\geq \inf_x q^{Z_{i-1}|X_{i-1}=x,Z_{1:(i-2)}}(z_{i-1}) \cdot \mathcal{L}_\theta^{X_i|Z_{1:(i-2)}}(x_i).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\dot{\mathcal{L}}_\theta^{X_i,Z_{i-1}|Z_{1:(i-2)}}(x_i, z_{i-1}) \\
&= \int q^{Z_{i-1}|X_{i-1},Z_{1:(i-2)}}(z_{i-1}) \left(\left[\dot{\mathcal{L}}_\theta^{X_i,X_{i-1}|Z_{1:(i-2)}}(x_i, x_{i-1}) \right]_+ - \left[\dot{\mathcal{L}}_\theta^{X_i,X_{i-1}|Z_{1:(i-2)}}(x_i, x_{i-1}) \right]_- \right) \mathrm{d}x_{i-1} \\
&\leq \sup_x q^{Z_{i-1}|X_{i-1}=x,Z_{1:(i-2)}}(z_{i-1}) \cdot \int \left[\dot{\mathcal{L}}_\theta^{X_i,X_{i-1}|Z_{1:(i-2)}}(x_i, x_{i-1}) \right]_+ \mathrm{d}x_{i-1}
\end{aligned}$$

$$- \inf_x q^{Z_{i-1}|X_{i-1}=x, Z_{1:(i-2)}}(z_{i-1}) \cdot \int \left[\dot{\mathcal{L}}_\theta^{X_i, X_{i-1}|Z_{1:(i-2)}}(x_i, x_{i-1})\right]_- \mathrm{d}x_{i-1}.$$

Since $\int \dot{\mathcal{L}}_\theta^{X_i, X_{i-1}|Z_{1:(i-2)}}(x_i, x_{i-1}) \, \mathrm{d}x_{i-1} = \dot{\mathcal{L}}_\theta^{X_i|Z_{1:(i-2)}}(x_i)$, we get

$$\int \left[\dot{\mathcal{L}}_\theta^{X_i, X_{i-1}|Z_{1:(i-2)}}(x_i, x_{i-1})\right]_+ \mathrm{d}x_{i-1} = \frac{1}{2} \int \left|\dot{\mathcal{L}}_\theta^{X_i, X_{i-1}|Z_{1:(i-2)}}(x_i, x_{i-1})\right| \mathrm{d}x_{i-1} + \frac{1}{2} \dot{\mathcal{L}}_\theta^{X_i|Z_{1:(i-2)}}(x_i)$$

$$\int \left[\dot{\mathcal{L}}_\theta^{X_i, X_{i-1}|Z_{1:(i-2)}}(x_i, x_{i-1})\right]_- \mathrm{d}x_{i-1} = \frac{1}{2} \int \left|\dot{\mathcal{L}}_\theta^{X_i, X_{i-1}|Z_{1:(i-2)}}(x_i, x_{i-1})\right| \mathrm{d}x_{i-1} - \frac{1}{2} \dot{\mathcal{L}}_\theta^{X_i|Z_{1:(i-2)}}(x_i).$$

Therefore,

$$\begin{aligned}
&\dot{\mathcal{L}}_\theta^{X_i, Z_{i-1}|Z_{1:(i-2)}}(x_i, z_{i-1}) \\
&\leq (e^\alpha - 1) \cdot (2 \wedge e^\alpha) \cdot \inf_x q^{Z_{i-1}|X_{i-1}=x, Z_{1:(i-2)}}(z_{i-1}) \cdot \frac{1}{2} \int \left|\dot{\mathcal{L}}_\theta^{X_i, X_{i-1}|Z_{1:(i-2)}}(x_i, x_{i-1})\right| \mathrm{d}x_{i-1} \\
&+ \frac{1}{2} \left(\sup_x q^{Z_{i-1}|X_{i-1}=x, Z_{1:(i-2)}}(z_{i-1}) + \inf_x q^{Z_{i-1}|X_{i-1}=x, Z_{1:(i-2)}}(z_{i-1}) \right) \left|\dot{\mathcal{L}}_\theta^{X_i|Z_{1:(i-2)}}(x_i)\right| \\
&\leq (e^\alpha - 1) \cdot \inf_x q^{Z_{i-1}|X_{i-1}=x, Z_{1:(i-2)}}(z_{i-1}) \cdot \int \left|\dot{\mathcal{L}}_\theta^{X_i, X_{i-1}|Z_{1:(i-2)}}(x_i, x_{i-1})\right| \mathrm{d}x_{i-1} \\
&+ \frac{1}{2}(e^\alpha + 1) \inf_x q^{Z_{i-1}|X_{i-1}=x, Z_{1:(i-2)}}(z_{i-1}) \left|\dot{\mathcal{L}}_\theta^{X_i|Z_{1:(i-2)}}(x_i)\right|. \qquad (38)
\end{aligned}$$

Thus, using that $(a+b)^2 \leq (1 + \frac{1}{\epsilon})a^2 + (1+\epsilon)b^2$ for any $\epsilon > 0$, we get that

$$\begin{aligned}
&\mathbb{E}_{Z_{1:(i-2)}} \left[\int \frac{(\dot{\mathcal{L}}_\theta^{X_i, Z_{i-1}|Z_{1:(i-2)}}(x_i, z_{i-1}))^2}{\mathcal{L}_\theta^{X_i, Z_{i-1}|Z_{1:(i-2)}}(x_i, z_{i-1})} \mathrm{d}x_i \, \mathrm{d}z_{i-1} \right] \\
&\leq \mathbb{E}_{Z_{1:(i-2)}} \Bigg[\int \inf_x q^{Z_{i-1}|X_{i-1}=x, Z_{1:(i-2)}}(z_{i-1}) \, \mathrm{d}z_{i-1} \\
&\Bigg((1 + \frac{1}{\epsilon})(e^\alpha - 1)^2 \int \frac{(\int \left|\dot{\mathcal{L}}_\theta^{X_i, X_{i-1}|Z_{1:(i-2)}}(x_i, x_{i-1})\right| dx_{i-1})^2}{\mathcal{L}_\theta^{X_i|Z_{1:(i-2)}}(x_i)} \mathrm{d}x_i \\
&+ (1+\varepsilon) \frac{1}{4}(e^\alpha + 1)^2 \int \frac{\left(\dot{\mathcal{L}}_\theta^{X_i|Z_{1:(i-2)}}(x_i)\right)^2}{\mathcal{L}_\theta^{X_i|Z_{1:(i-2)}}(x_i)} \mathrm{d}x_i \Bigg) \Bigg].
\end{aligned}$$

Note first that $\mathbb{E}_{Z_{1:(i-2)}} \left[\int \frac{\left(\dot{\mathcal{L}}_\theta^{X_i|Z_{1:(i-2)}}(x_i)\right)^2}{\mathcal{L}_\theta^{X_i|Z_{1:(i-2)}}(x_i)} \mathrm{d}x_i \right]$ is the Fisher information $I(X_i|Z_{1:(i-2)})$ and then

$$\begin{aligned}
&\mathbb{E}_{Z_{1:(i-2)}} \left[\int \frac{(\int \left|\dot{\mathcal{L}}_\theta^{X_i, X_{i-1}|Z_{1:(i-2)}}(x_i, x_{i-1})\right| \mathrm{d}x_{i-1})^2}{\mathcal{L}_\theta^{X_i|Z_{1:(i-2)}}(x_i)} \mathrm{d}x_i \right] \\
&\leq \mathbb{E}_{Z_{1:(i-2)}} \left[\int \int \frac{\left|\dot{\mathcal{L}}_\theta^{X_i, X_{i-1}|Z_{1:(i-2)}}(x_i, x_{i-1})\right|^2}{\mathcal{L}_\theta^{X_i, X_{i-1}|Z_{1:(i-2)}}(x_i)} \mathrm{d}x_{i-1} \, \mathrm{d}x_i \right]
\end{aligned}$$

which is the Fisher information $I(X_i, X_{i-1}|Z_{1:(i-2)})$ and this finishes the proof of (10).

For $i \geq 2$, we prove (11), that is

$$I(X_i|Z_{1:(i-1)}) \leq (1 + \frac{9}{4}(e^\alpha - 1)^2) \left(I(X_i, X_{i-1}|Z_{1:(i-2)}) + I(X_i|Z_{1:(i-2)}) \right).$$

Let us go back to (38). We decompose $\frac{1}{2}(e^\alpha + 1) = \frac{1}{2}(e^\alpha - 1) + 1$ and use that:

$$\left| \dot{\mathcal{L}}_\theta^{X_i|Z_{1:(i-2)}}(x_i) \right| = \left| \int \dot{\mathcal{L}}_\theta^{X_i, X_{i-1}|Z_{1:(i-2)}}(x_i, x_{i-1}) \, \mathrm{d}x_{i-1} \right| \leq \int \left| \dot{\mathcal{L}}_\theta^{X_i, X_{i-1}|Z_{1:(i-2)}}(x_i, x_{i-1}) \right| \, \mathrm{d}x_{i-1}.$$

Therefore,

$$\begin{aligned}
&\dot{\mathcal{L}}_\theta^{X_i, Z_{i-1}|Z_{1:(i-2)}}(x_i, z_{i-1}) \\
&\leq \frac{3}{2}(e^\alpha - 1) \cdot \inf_x q^{Z_{i-1}|X_{i-1}=x, Z_{1:(i-2)}}(z_{i-1}) \cdot \int \left| \dot{\mathcal{L}}_\theta^{X_i, X_{i-1}|Z_{1:(i-2)}}(x_i, x_{i-1}) \right| \, \mathrm{d}x_{i-1} \\
&+ \inf_x q^{Z_{i-1}|X_{i-1}=x, Z_{1:(i-2)}}(z_{i-1}) \left| \dot{\mathcal{L}}_\theta^{X_i|Z_{1:(i-2)}}(x_i) \right|.
\end{aligned}$$

Similarly to the previous proof we obtain for an arbitrary $\epsilon > 0$ that

$$I(X_i|Z_{1:(i-1)}) \leq (1 + \frac{1}{\epsilon})\frac{9}{4}(e^\alpha - 1)^2 I(X_i, X_{i-1}|Z_{1:(i-2)}) + (1 + \epsilon) I(X_i|Z_{1:(i-2)})$$

and conclude by choosing $\epsilon = \frac{9}{4}(e^\alpha - 1)^2$. ■

A.10 Proof of Theorem 10

Let us consider a centered Gaussian time series. Define the parametric family of covariance matrices $\Sigma(\theta)$, where θ belongs to $[-T, T]$ for some given $T > 0$, such that, for some constant $\sigma > 0$

$$\sigma_0 = 1; \, \sigma_j = 0, \text{ for } j \notin \{0, K\} \quad \text{and} \quad \sigma_K(\theta) = \theta \cdot \sigma \cos(K w_0), \; K \geq 1.$$

We assume that we find w_0 in $(-\pi, \pi)$ such that $\cos(K w_0) \neq 0$. These stationary Gaussian processes are associated to the parametric family of spectral densities

$$\begin{aligned}
f_\theta(w) &= \frac{\sigma_0}{2\pi} + \frac{\sigma_K(\theta)}{\pi} \cdot \cos(Kw), \\
&= \frac{\sigma_0}{2\pi} + \theta \frac{\sigma \cos(K w_0)}{\pi} \cdot \cos(Kw) \\
&= \frac{\sigma_0}{2\pi} + \theta \frac{\sigma}{2\pi} \cdot \left[\cos(K(w - w_0)) + \cos(K(w + w_0)) \right].
\end{aligned}$$

Indeed, f_θ is still a symmetric function and bounded from below by some positive constant

$$\frac{\sigma_0}{2\pi} - T\frac{\sigma}{\pi} \geq c > 0,$$

for any θ in the set $[-T, T]$ and properly chosen T.

Let us see that estimating $\sigma_K(\theta)$ can be reduced to estimating $f_\theta(w_0)$ in our model, since

$$\sigma_K(\theta) = \left(f_\theta(w_0) - \frac{\sigma_0}{2\pi} \right) \cdot \frac{\pi}{\cos(K w_0)}$$

and σ_0 is known. Thus, we bound the minimax risk from below as follows:

$$R := \inf_{\hat{\sigma}_K} \sup_{Q\in\mathcal{Q}_\alpha^{\mathrm{NI}}} \sup_{\sigma:\|\sigma\|_2^2\leq M_1} \mathbb{E}_\theta\left[|\hat{\sigma}_K - \sigma_K|^2\right] \geq \inf_{\hat{\sigma}_K} \sup_{Q\in\mathcal{Q}_\alpha^{\mathrm{NI}}} \sup_{\theta\in[-T,T]} \mathbb{E}_\theta\left[|\hat{\sigma}_K - \sigma_K(\theta)|^2\right]$$
$$\gtrsim \inf_{\hat{f}_n} \sup_{Q\in\mathcal{Q}_\alpha^{\mathrm{NI}}} \sup_{\theta\in[-T,T]} \mathbb{E}_\theta\left[|\hat{f}_n(w_0) - f_\theta(w_0)|^2\right]$$

by first reducing the risk over all possible covariance matrices Σ (uniformly bounded in spectral norm) to the parametric family $\Sigma(\theta)$. Then we changed the problem of recovering $\sigma_K(\theta)$ into that of recovering f_θ. Next, we reduce the problem to a Bayesian problem:

$$R \gtrsim \inf_{\hat{f}_n} \sup_{Q\in\mathcal{Q}_\alpha^{\mathrm{NI}}} \int_{[-T,T]} \mathbb{E}_\theta\left[|\hat{f}_n(w_0) - f_\theta(w_0)|^2\right] \mathrm{d}\lambda_T(\theta),$$

where λ is an a priori measure on [-1,1] with finite Fisher information

$$I(\lambda) := \int_{[-1,1]} (\lambda'(u))^2/\lambda(u)\,\mathrm{d}u$$

and λ_T is λ re-scaled to the interval $[-T,T]$. Thus, $I(\lambda_T) = I(\lambda)/T^2$.
We apply the van Trees inequality to get

$$\int_{[-T,T]} \mathbb{E}_\theta\left[|\hat{f}_n(w_0) - f_\theta(w_0)|^2\right] \mathrm{d}\lambda_T(\theta) \geq \frac{(\int_{[-T,T]} \frac{\partial}{\partial\theta} f_\theta(w_0)\,\mathrm{d}\lambda_T(\theta))^2}{\int_{[-T,T]} I(Z_{1:n})\,\mathrm{d}\lambda_T(\theta) + I(\lambda_T)}$$
$$\geq \frac{(\int_{[-T,T]} \sigma\cos^2(Kw_0)/\pi\,\mathrm{d}\lambda_T(\theta))^2}{\int_{[-T,T]} I(Z_{1:n})\,\mathrm{d}\lambda_T(\theta) + I(\lambda)/T^2}$$
$$\geq \frac{\sigma^2\cos^4(Kw_0)/\pi^2}{\int_{[-T,T]} I(Z_{1:n})\,\mathrm{d}\lambda_T(\theta) + I(\lambda)/T^2}.$$

Let $(Z_1, ..., Z_n)$ be the time series obtained by a non-interactive $\alpha-$differentially private mechanism, Q in $\mathcal{Q}_\alpha^{\mathrm{NI}}$. The corresponding likelihoods of the vectors $X_{1:n}$ and of $Z_{1:n}$ are denoted by $\mathcal{L}_\theta^{X_{1:n}}$ and $\mathcal{L}_\theta^{Z_{1:n}}$, respectively.

We want to give an upper bound for $I(Z_{1:n})$ in order to finish the proof. Let us note that by our choice of the covariance matrix $\Sigma(\theta)$ the likelihood $\mathcal{L}_\theta^{X_{1:n}}$ can be decomposed as a product of K likelihoods :

$$\mathcal{L}_\theta^{X_n, X_{n-K},\ldots,X_{n-N_0K}} \cdot \mathcal{L}_\theta^{X_{n-1},X_{n-1-K},\ldots,X_{n-1-N_1K}} \cdot\ldots\cdot \mathcal{L}_\theta^{X_{n-K+1},X_{n-2K+1},\ldots,X_{n-N_{K-1}K+1}},$$

where $N_i = \lfloor (n-i-K)/K \rfloor$ for i from 0 to $K-1$. Each vector is 1-dependent. The non-interactive privacy mechanism preserves this structure, so the same holds for $\mathcal{L}_\theta^{Z_{1:n}}$. Since the Fisher information of jointly independent vectors is additive, we write :

$$I(Z_{1:n}) = \sum_{i=0}^{K-1} I(Z_{n-i}, Z_{n-i-K}, \ldots, Z_{n-i-N_iK}). \tag{39}$$

We shall first give the proof for the case $K=1$ when the whole vector $X_{1:n}$ is 1-dependent. Next, we will adapt the calculations to the case $K>1$ using the decomposition above.

Let us suppose $K = 1$. First we apply (9) and next (10) in Lemma 9 to get:

$$
\begin{aligned}
I(Z_{1:n}) &= \sum_{i=1}^{n} I(Z_i | Z_{1:(i-1)}) \\
&\lesssim (e^{\alpha} - 1)^2 \sum_{i=1}^{n} I(X_i | Z_{1:(i-1)}) \qquad (40) \\
&\lesssim (e^{\alpha} - 1)^2 \sum_{i=1}^{n} (A \cdot I(X_i, X_{i-1} | Z_{1:(i-2)}) + B \cdot I(X_i | Z_{1:(i-2)})),
\end{aligned}
$$

with $A = (1 + \frac{1}{\epsilon})(e^{\alpha} - 1)^2$ and $B = (1 + \epsilon)e^{2\alpha}$ for an arbitrary $\epsilon > 0$ such that $A < 1$. Since X_i is independent of $X_{1:(i-2)}$, then we can easily show that X_i is independent of $Z_{1:(i-2)}$. Thus $I(X_i | Z_{1:(i-2)})) = I(X_i)$. By iterating and repeating this reasoning we get that:

$$
I(X_i | Z_{1:(i-1)}) \leq A^{i-1} I(X_{1:i}) + A^{i-2} B I(X_{2:i}) + \ldots + AB I(X_{i-1}, X_i) + B I(X_i). \qquad (41)
$$

Note that X_i has a likelihood free of the parameter θ so it does not bring any information on θ, that is $I(X_i) = 0$.

Let us prove now that $I(X_{i-\ell}, \ldots, X_i) \leq C\ell\sigma^2$, for any ℓ from 1 to $i - 1$, where $C = (4\pi^2 c^2)^{-1}$ depends on c introduced in the definition of the spectral density f_θ. The vector $X_{(i-\ell):i}$ has a centered Gaussian distribution with covariance matrix $\Sigma_1^{\ell+1}(\theta)$ given by:

$$
\Sigma_1^{\ell+1}(\theta) = \sigma_0 \cdot \mathbb{I}_{\ell+1} + \sigma_1(\theta) \cdot M, \quad [M]_{i,j} = \mathbf{1}(|i - j| = 1),
$$

where $\mathbb{I}_{\ell+1}$ denotes the identity matrix of size $\ell + 1$. Let us denote by $u_0, \ldots, u_\ell$ the eigenvalues of the matrix M. Note that M and $\Sigma_1^{\ell+1}(\theta)$ share the same eigenvectors. We can write:

$$
\begin{aligned}
I(X_{i-\ell}, \ldots, X_i) &= \frac{1}{2} tr \left[\left(\Sigma_1^{\ell+1}(\theta)^{-1} \frac{\partial}{\partial\theta} \Sigma_1^{\ell+1}(\theta) \right)^2 \right] \\
&= \frac{1}{2} \left\| \Sigma_1^{\ell+1}(\theta)^{-1} \cdot \sigma \cos(w_0) M \right\|_F^2 \\
&= \frac{1}{2} \sum_{k=0}^{\ell} \frac{\sigma^2 \cos^2(w_0) u_k^2}{(\sigma_0 + \sigma_1(\theta) \cdot u_k)^2} \\
&\leq \frac{\sigma^2 \cos^2(w_0) \|M\|_F^2}{2(\sigma_0 - T\sigma \|M\|_\infty)^2}.
\end{aligned}
$$

It is easy to calculate that $\|M\|_F^2 = 2\ell$ and that its operator norm is $\|M\|_\infty = 2\cos(\pi/(\ell + 2)) \leq 2$. By our construction of the spectral density we get that:

$$
I(X_{i-\ell}, \ldots, X_i) \leq \frac{\sigma^2 \cdot 2\ell}{2(2\pi c)^2} \leq \frac{\sigma^2}{4\pi^2 c^2} \ell = C\ell\sigma^2.
$$

Use this in (41) to get:

$$
I(X_i | Z_{1:(i-1)}) \leq AB \sum_{\ell=1}^{i-1} C\ell\sigma^2 A^{\ell-1} \lesssim (e^{\alpha} - 1)^2 \frac{BC\sigma^2}{(1 - A)^2}.
$$

Finally, we use this in (40) to get:

$$I(Z_{1:n}) \lesssim n(e^{\alpha}-1)^4\sigma^2,$$

up to constants depending on fixed positive constants ϵ and c. This finishes the proof for the case $K=1$.

Now, for $K>1$ and each i in $0,\dots,K-1$, let us bound similarly each term in (39), of the form $I(Z_{n-i}, Z_{n-i-K}, ..., Z_{n-i-N_iK})$. We get

$$\begin{aligned}
& I(Z_{n-i}, Z_{n-i-K}, ..., Z_{n-i-N_iK}) \\
= \; & I(Z_{n-i}|Z_{n-i-K}, ..., Z_{n-i-N_iK}) + I(Z_{n-i-K}|Z_{n-i-2K}, ..., Z_{n-i-N_iK}) + \dots + I(Z_{n-i-N_iK}) \\
\lesssim \; & (e^{\alpha}-1)^2 (I(X_{n-i}|Z_{n-i-K}, ..., Z_{n-i-N_iK}) + I(X_{n-i-K}|Z_{n-i-2K}, ..., Z_{n-i-N_iK}) \\
& + \dots + I(X_{n-i-N_iK})).
\end{aligned}$$

We iterate the procedure by applying (10) in Lemma 9 in order to get geometric series analogous to (41). Recall that the subvector $X_{n-i}, X_{n-i-K}, ..., X_{n-i-\ell K}$ has a centered Gaussian distribution with covariance matrix

$$\Sigma_K^{\ell+1}(\theta) = \sigma_0 \cdot \mathbb{I}_{\ell+1} + \sigma_K(\theta)\cdot M, \quad \ell = 1, ..., N_i.$$

Following the same steps as for $K=1$ we get:

$$\begin{aligned}
I(X_{n-i-\ell K}, \dots, X_{n-i}) &= \frac{1}{2} tr\left[\left(\Sigma_K^{\ell+1}(\theta)^{-1}\frac{\partial}{\partial\theta}\Sigma_K^{\ell+1}(\theta)\right)^2\right] = \frac{1}{2}\left\|\Sigma_K^{\ell+1}(\theta)^{-1}\cdot\sigma\cos(Kw_0)M\right\|_F^2 \\
&= \frac{1}{2}\sum_{k=0}^{\ell}\frac{\sigma^2\cos^2(Kw_0)u_k^2}{(\sigma_0+\sigma_K(\theta)\cdot u_k)^2} \\
&\le \frac{\sigma^2\cos^2(Kw_0)\|M\|_F^2}{2(\sigma_0 - T\sigma\|M\|_\infty)^2} \le \frac{\sigma^2\ell}{(\sigma_0-2T\sigma)^2} \le \frac{\sigma^2\ell}{4\pi^2c^2},
\end{aligned}$$

where $c>0$ is chosen in the definition of the spectral density f_θ. Further we get that:

$$I(X_{n-i}|Z_{n-i-K}, ..., Z_{n-i-N_iK}) \lesssim (e^{\alpha}-1)^2\frac{BC\sigma^2}{(1-A)^2}$$

and, finally, since $N_i \le n/K$:

$$I(Z_{1:n}) \;\lesssim\; \sum_{i=0}^{K-1}\frac{n}{K}(e^{\alpha}-1)^4\sigma^2 = n(e^{\alpha}-1)^4\sigma^2,$$

up to constants depending on fixed positive constants ϵ and c.

Finally,

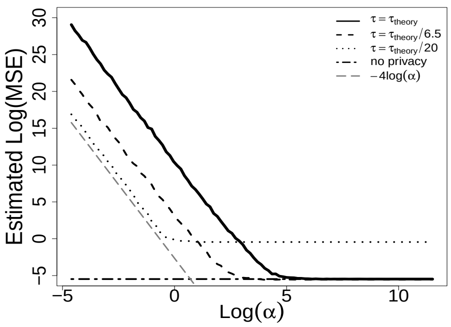
$$I(Z_{1:n}) \lesssim (e^{\alpha}-1)^2\sum_{i=1}^{n} I(X_i|Z_{1:(i-1)}) \lesssim n(e^{\alpha}-1)^4\sigma^2.$$

We conclude by recalling that w_0 was chosen such that $\cos(Kw_0)\neq 0$ and by plugging this into the Van Trees inequality above, to get that, for large enough n,

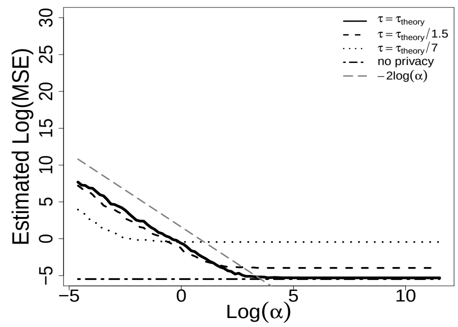
$$\inf_{\hat\sigma_K}\sup_{Q\in\mathcal{Q}_\alpha^{\mathrm{NI}}}\sup_{\sigma:\|\sigma\|_2^2\le M_1}\mathbb{E}_\theta\left[|\hat\sigma_K-\sigma_K|^2\right] \gtrsim \frac{\sigma^2\cos^4(Kw_0)/\pi^2}{n(e^{\alpha}-1)^4\sigma^2 + I(\lambda)/T^2} \asymp \frac{1}{n\alpha^4}.$$

■

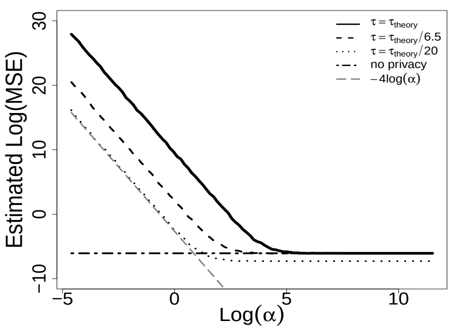
Appendix B. Additional Simulation Results



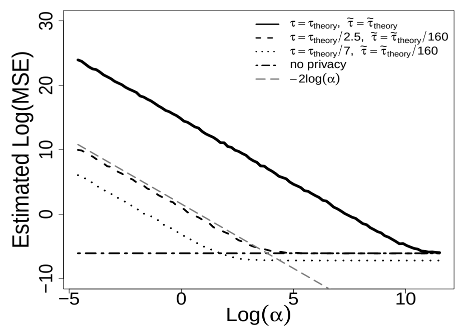
(a) Estimation of σ_0=1.44 from NI-privatized data.



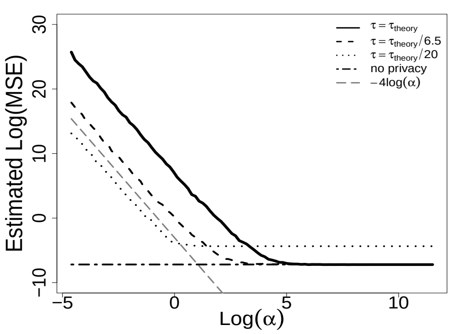
(b) Estimation of σ_0=1.44 from SI-privatized data.



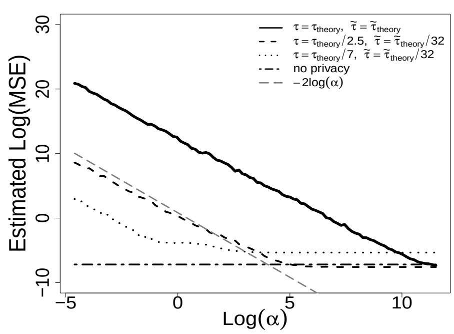
(c) Estimation of σ_2=0.02 from NI-privatized data.



(d) Estimation of σ_2=0.02 from SI-privatized data.

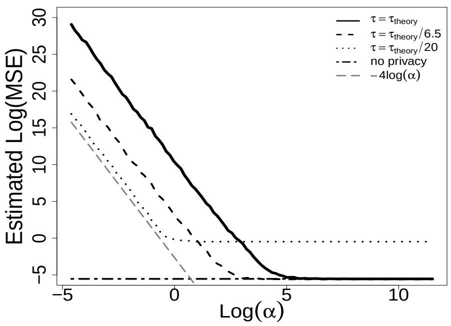


(e) Estimation of $f(\pi/5)$=0.19 from NI-privatized data.

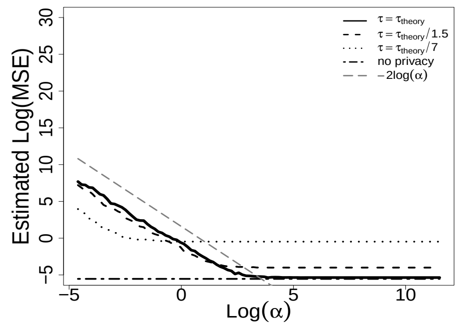


(f) Estimation of $f(\pi/5)$=0.19 from SI-privatized data.

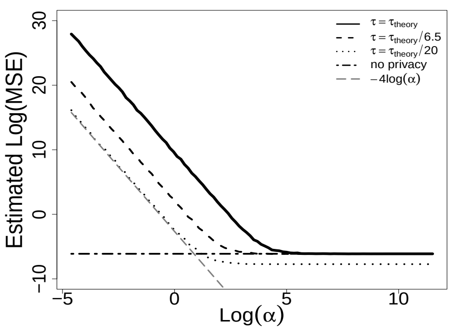
Figure 2: Example (2): Log-log plot of the estimated mean squared error (MSE) versus the privacy level α for estimation of variance (top), covariance coefficient (center) and pointwise spectral density (bottom) using NI-privatized data (left) and SI-privatized data (right). Gray dashed lines indicate the theoretical α-scaling of the MSE for NI- and SI-based estimators, including $\log(n)$-factors.



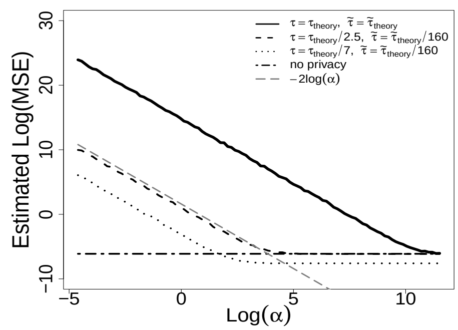
(a) Estimation of σ_0=1.44 from NI-privatized data.



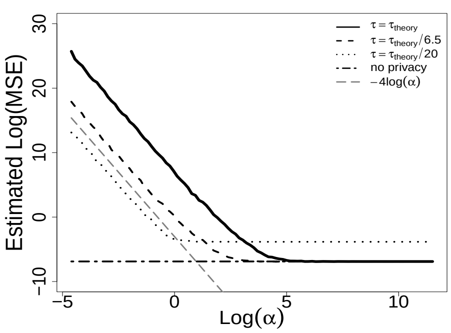
(b) Estimation of σ_0=1.44 from SI-privatized data.



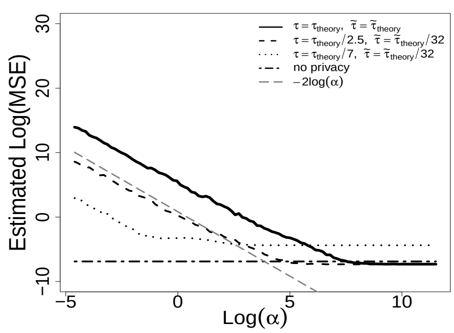
(c) Estimation of σ_2=0.005 from NI-privatized data.



(d) Estimation of σ_2=0.005 from SI-privatized data.



(e) Estimation of $f(\pi/5)$=0.24 from NI-privatized data.



(f) Estimation of $f(\pi/5)$=0.24 from SI-privatized data.

Figure 3: Example (3): Log-log plot of the estimated mean squared error (MSE) versus the privacy level α for estimation of variance (top), covariance coefficient (center) and pointwise spectral density (bottom) using NI-privatized data (left) and SI-privatized data (right). Gray dashed lines indicate the theoretical α-scaling of the MSE for NI- and SI-based estimators, including $\log(n)$-factors.

We report the estimated mean squared error (MSE) for private estimation of σ_0, σ_2, and $f(\pi/5)$ from centered Gaussian stationary processes with a 0.8-Hölder continuous spectral density (Example 2) and with polynomially decaying covariances $\sigma_k = 1.44(1+|k|)^{-5.1}$ (Example 3), using both sequentially interactive (SI) and non-interactive (NI) mechanisms. Tuning parameters and simulation settings are chosen as described in Section 4. The corresponding results are shown in Figure 2 for Example (2) and Figure 3 for Example (3).

References

Kareem Amin, Travis Dick, Alex Kulesza, Andres Munoz, and Sergei Vassilvitskii. Differentially private covariance estimation. *Advances in Neural Information Processing Systems*, 32, 2019.

Cristina Butucea, Amandine Dubois, and Adrien Saumard. Phase transitions for support recovery under local differential privacy. *Mathematical Statistics and Learning*, 6(1):1–50, 2023.

Yang Cao, Masatoshi Yoshikawa, Yonghui Xiao, and Li Xiong. Quantifying differential privacy in continuous data release under temporal correlations. *IEEE Transactions on Knowledge and Data Engineering*, 31(7):1281–1295, 2018.

John C Duchi, Michael I Jordan, and Martin J Wainwright. Minimax optimal procedures for locally private estimation. *Journal of the American Statistical Association*, 113(521): 182–201, 2018.

Cynthia Dwork. Differential privacy. In *International Colloquium on Automata, Languages, and Programming*, pages 1–12. Springer, 2006.

Leo A Goodman. Snowball sampling. *The Annals of Mathematical Statistics*, pages 148–170, 1961.

Yitzhak Katznelson. *An Introduction to Harmonic Analysis*. Cambridge University Press, 2004.

Daniel Kifer and Ashwin Machanavajjhala. Pufferfish: A framework for mathematical privacy definitions. *ACM Transactions on Database Systems (TODS)*, 39(1):1–36, 2014.

Martin Kroll. Nonparametric spectral density estimation under local differential privacy. *Statistical Inference for Stochastic Processes*, pages 1–35, 2024.

Franklin Leukam Lako, Paul Lajoie-Mazenc, and Maryline Laurent. Privacy-preserving publication of time-series data in smart grid. *Security and Communication Networks*, 2021(1):6643566, 2021.

Lingjuan Lyu, Yee Wei Law, Jiong Jin, and Marimuthu Palaniswami. Privacy-preserving aggregation of smart metering via transformation and encryption. In *2017 IEEE Trustcom/BigDataSE/ICESS*, pages 472–479. IEEE, 2017.

Tucker McElroy, Anindya Roy, and Gaurab Hore. Flip: A utility preserving privacy mechanism for time series. *Journal of Machine Learning Research*, 24(111):1–29, 2023.

Vibhor Rastogi and Suman Nath. Differentially private aggregation of distributed time-series with transformation and encryption. In *2010 ACM International Conference on Management of Data*, pages 735–746, 2010.

Yashothara Shanmugarasa, MAP Chamikara, Hye-young Paik, Salil S Kanhere, and Liming Zhu. Local differential privacy for smart meter data sharing with energy disaggregation. In *2024 International Conference on Distributed Computing in Smart Systems and the Internet of Things (DCOSS-IoT)*, pages 1–10. IEEE, 2024.

Nazar Waheed, Fazlullah Khan, Spyridon Mastorakis, Mian Ahmad Jan, Abeer Z Alalmaie, and Priyadarsi Nanda. Privacy-enhanced living: A local differential privacy approach to secure smart home data. In *2023 IEEE International Conference on Omni-layer Intelligent Systems (COINS)*, pages 1–6. IEEE, 2023.

Di Wang and Jinhui Xu. Tight lower bound of sparse covariance matrix estimation in the local differential privacy model. *Theoretical Computer Science*, 815:47–59, 2020.

Di Wang and Jinhui Xu. Differentially private high dimensional sparse covariance matrix estimation. *Theoretical Computer Science*, 865:119–130, 2021.

Hao Wang and Zhengquan Xu. Cts-dp: Publishing correlated time-series data via differential privacy. *Knowledge-Based Systems*, 122:167–179, 2017.

Stanley L Warner. Randomized response: A survey technique for eliminating evasive answer bias. *Journal of the American Statistical Association*, 60(309):63–69, 1965.

Yonghui Xiao and Li Xiong. Protecting locations with differential privacy under temporal correlations. In *22nd ACM Conference on Computer and Communications Security*, pages 1298–1309, 2015.

Bin Yang, Issei Sato, and Hiroshi Nakagawa. Bayesian differential privacy on correlated data. In *2015 ACM international conference on Management of Data*, pages 747–762, 2015.

Tao Zhang, Tianqing Zhu, Renping Liu, and Wanlei Zhou. Correlated data in differential privacy: definition and analysis. *Concurrency and Computation: Practice and Experience*, 34(16):e6015, 2022a.

Xueru Zhang, Mohammad Mahdi Khalili, and Mingyan Liu. Differentially private real-time release of sequential data. *ACM Transactions on Privacy and Security*, 26(1):1–29, 2022b.

Tianqing Zhu, Ping Xiong, Gang Li, and Wanlei Zhou. Correlated differential privacy: hiding information in non-iid data set. *IEEE Transactions on Information Forensics and Security*, 10(2):229–242, 2014.

Antoni Zygmund. *Trigonometric Series*, volume 1. Cambridge university press, 2002.