7. Proofs

7.1. Proof of Theorem 1

Observe that $Y^* \in \mathcal{S}_{\text{md}} \subset \mathcal{S}_{\text{nuclear}}$, so it suffices to prove the theorem assuming $\mathcal{S} = \mathcal{S}_{\text{nuclear}}$ in (3).

We need some additional notation. Suppose the size of the $i$-th cluster is $K_i$, and the rank-$r$ SVD of $Y^*$ is $U \Sigma U^T$. Note that $UU^T$ is a block diagonal matrix with $r$ blocks such that the $i$-th block has size $K_i \times K_i$ with all entries equal to $\frac{1}{K_i} \leq \frac{1}{K}$. We define the projections $P_T$ and $P_{T^\perp}$ by

$$P_T Z = UU^T Z + ZUU^T - UU^T ZU$$

and

$$P_{T^\perp} Z = Z - P_T Z.$$

Define the matrix $W_{ij} := ((2A_{ij} - 1)B_{ij})_{i,j=1}^n \in \mathbb{R}^{n \times n}$ and the quantities $\theta := E[(1 - 2P_{ij})B_{ij}]$ and $\rho := E[B_{ij}^2] - \theta^2$.

Note that

$$\text{EW}_{ij} = E[2A_{ij} - 1)B_{ij}|P] = (2Y_{ij}^* - 1)E[(1 - 2P_{ij})B_{ij}] = (2Y_{ij}^* - 1)\theta$$

and

$$\text{Var}[W_{ij}] = E[W_{ij}^2] - (\text{E}W_{ij})^2 = E[(2A_{ij} - 1)^2B_{ij}^2] - (2Y_{ij}^* - 1)^2 \theta^2 = \rho.$$

Our proof requires two standard concentration results for the random matrix $W$.

**Lemma 1.** If $0 \leq W_{ij} \leq b_0$ almost surely for all $i, j$ and the condition (6) holds, then with high probability, we have

$$||W - E[W]|| \leq c_2 \left(b \log n + \sqrt{\rho n \log n}\right)$$

and

$$||UU^T (W - E[W])||_\infty \leq c_3 \frac{\sqrt{b^2 \log \rho n} + \rho K \log n}{K}$$

for some universal constants $c_2$, $c_3$.

We prove the lemma in Section 7.1.1 to follow. We now prove Theorem 1 assuming the two inequalities (12) and (13) in the lemma hold.

For any matrix $Y$, we define $\Delta(Y) := \langle Y^* - Y, W \rangle$. To prove the theorem, it suffices to show that $\Delta(Y) > 0$ for all feasible $Y$ of the program 2–4 with $Y \neq Y^*$. We rewrite $\Delta(Y)$ as

$$\Delta(Y) = \langle EW, Y^* - Y \rangle + \langle W - EW, Y^* - Y \rangle.$$  

(14)

We bound the two terms above. For any feasible $Y$ obeying the constraint 4, the first term in (14) can be written as

$$\langle EW, Y^* - Y \rangle = \sum_{i,j} (2Y_{ij}^* - 1) \theta \cdot (Y_{ij}^* - Y_{ij})$$

$$= \theta ||Y^* - Y||_1,$$

where the last equality follows from $0 \leq Y_{ij} \leq 1, \forall i, j$.

On the other hand, if we let $\lambda := c_2 \left( \log n + \sqrt{\rho n \log n} \right)$, then by (12) we have

$$\left\| \frac{1}{\lambda} P_{T^\perp} (W - EW) \right\| \leq \left\| \frac{1}{\lambda} (W - EW) \right\| \leq 1.$$

This means $UU^T + \frac{1}{\lambda} P_{T^\perp} (W - EW)$ is a subgradient of the function $f(X) = ||X||_*$ at $X = Y^*$. Therefore, for any feasible $Y$ we have

$$0 \geq ||Y^*||_* - ||Y||_* \geq \langle UU^T + \frac{1}{\lambda} P_{T^\perp} (W - EW), Y - Y^* \rangle,$$

which means

$$\langle W - EW, Y^* - Y \rangle \geq \langle P_T (W - EW) - \lambda UU^T, Y^* - Y \rangle$$

(16)
We substitute (15) and (16) into (14) to obtain that for all feasible \( Y \),
\[
\Delta(Y) \geq \theta \|Y^* - Y\|_1 + \langle \mathcal{P}_T (W - EW) - \lambda UU^T, Y^* - Y \rangle
\]
\[
\overset{(a)}{\geq} \left( \theta - \lambda \|UU^T\|_{\infty} - \|\mathcal{P}_T (W - EW)\|_{\infty} \right) \|Y^* - Y\|_1
\]
\[
\overset{(b)}{\geq} \left( \theta - \frac{\lambda}{K} - \|\mathcal{P}_T (W - EW)\|_{\infty} \right) \|Y^* - Y\|_1,
\]
where (a) follows from the Holder’s inequality and (b) follows from the structure of \( U \). But by definition of \( \mathcal{P}_T \), we have
\[
\|\mathcal{P}_T (W - EW)\|_{\infty} \leq \|UU^T (W - EW)\|_{\infty} + \|(W - EW) UU^T\|_{\infty} + \|UU^T (W - EW) UU^T\|_{\infty}
\]
\[
\leq 3 \|UU^T (W - EW)\|_{\infty} \leq 3 c_3 \sqrt{b^2 \log^2 n + K \rho \log n} \frac{K}{K},
\]
where the last inequality follows from (13). It follows that
\[
\Delta(Y) \geq \left( \theta - c_2 \left( b \log n + \sqrt{\rho n \log n} \right) - 3 c_3 \sqrt{b^2 \log^2 n + K \rho \log n} \right) \|Y^* - Y\|_1.
\]

If the condition 6 in the theorem holds, then the quantity inside the parenthesis is positive (note that \( \rho \leq \mathbb{E} \left[ B^2_{i,j} \right] \)). This means \( \Delta(Y) > 0 \) for all \( Y \neq Y^* \), which proves the theorem.

7.1.1. Proof of Lemma 1

Let \( e_i \) be the \( i \)-th standard basis vector in \( \mathbb{R}^n \). For the first inequality in the lemma, note that
\[
W - EW = \sum_{i,j} (W_{ij} - EW_{ij}) e_i e_j^T,
\]
which is the sum of \( n^2 \) i.i.d. zero-mean matrix. We compute
\[
\| (W_{ij} - EW_{ij}) e_i e_j^T \| = |W_{ij} - EW_{ij}| \leq b
\]
for all \( (i, j) \) and
\[
\left\| \mathbb{E} \sum_{i,j} (W_{ij} - EW_{ij})^2 e_j e_i^T e_i e_j^T \right\| = \left\| \mathbb{E} \sum_{i,j} (W_{ij} - EW_{ij})^2 e_j e_i^T e_j e_i^T \right\|
\]
\[
= \rho \left\| e_i e_i^T \right\| = \rho n.
\]
Applying the matrix Bernstein inequality (Tropp, 2012) gives that w.h.p.
\[
\|W - EW\| \leq c_2 \left( b \log n + \sqrt{\rho n \log n} \right)
\]
for some constant \( c_2 \).

We prove the second inequality. Fix \( (i, j) \). Assume node \( i \) belongs to the cluster \( k \). Then
\[
(UU^T (W - EW))_{ij} = \frac{1}{K_k} \sum_{i' \in C_k} (W - EW)_{i'j},
\]
which is the average of \( K_k \) independent zero-mean random variables taking values in \([-b, b]\) with variance bounded by \( \rho \). Therefore, by standard Bernstein inequality, we know that for some constant \( c_3 \) that
\[
\left\| (UU^T (W - EW))_{ij} \right\| \leq \frac{1}{K_k} c_3 \left( b \log n + \sqrt{\rho K_k \log n} \right) \leq c_3 \sqrt{b^2 \log^2 n + K \rho \log n} \frac{K}{K}, \text{ w.h.p.}
\]
where the last inequality follows from \( K_k \geq K \). The lemma follows from a union bound over all \( (i, j) \).
where the first inequality follows from symmetry. Let 

\[ B_{ij} = B_{ij}^{\text{MLE}} = \log \frac{1 - P_{ij}}{P_{ij}} \leq \min \left\{ \frac{1 - 2P_{ij}}{P_{ij}}, \log \frac{1}{\epsilon} \right\} \leq 10 \left( 1 - 2P_{ij} \right) \log \frac{1}{\epsilon}, \]  

(17)

so \( B_{ij} \leq b_0 \). The condition (8) in the corollary statement implies that 

\[ \mathbb{E} \left[ \left( \frac{1}{2} - P_{ij} \right) B_{ij} \right] \geq c_1 \cdot \frac{n \log n}{K^2} \cdot \frac{b_0}{10} \geq c_1 \frac{b_0 \log n}{10} \]

since \( \bar{P}_{ij} \geq P_{ij} \) and \( K \leq n \). On the other hand, the second term in the RHS of (6) can be upper bounded as follows:

\[
c_0 \sqrt{\mathbb{E} \left[ B_{ij}^2 \right]} \cdot \frac{\sqrt{n \log n}}{K} = c_0 \frac{\sqrt{n \log n}}{K} - \left( 1 \right) 2 \mathbb{E} \left[ \log \frac{1 - P_{ij}}{P_{ij}} \right] \cdot \frac{\sqrt{n \log n}}{K} \\
\leq 10c_0 \frac{\sqrt{n \log (1/\epsilon) \log n}}{K} \\
\leq \frac{c_0}{2} \mathbb{E} \left[ \left( 1 - 2P_{ij} \right) \log \frac{1 - \bar{P}_{ij}}{P_{ij}} \right] \\
= \frac{c_0}{2} \mathbb{E} \left[ \left( 1 - 2P_{ij} \right) B_{ij} \right],
\]

where the inequality (a) follows from the previous bound (17), and (b) follows from the condition (8) in the corollary statement and \( \bar{P}_{ij} \geq P_{ij} \). Combining the last two display equations proves that (6) is satisfied.

For the second part of the corollary, we note that \( \bar{P}_{ij} := \max \left\{ \frac{1}{16}, P_{ij} \right\} \geq \epsilon := \frac{1}{16} \). The RHS of (8) is upper bounded by \( \log 16 \cdot c_1 \frac{n \log n}{K} \). Because \( \log \frac{1 - x}{x} \geq \frac{1}{10} \left( 1 - 2x \right) = \frac{1}{5} \left( \frac{1}{2} - x \right) \) for all \( x \leq \frac{1}{2} \), we have \( \frac{1}{5} \left( \frac{1}{2} - \bar{P}_{ij} \right)^2 \geq \frac{1}{16} \left( \frac{1}{2} - P_{ij} \right)^2 \) almost surely. It follows that the LHS of (8) is lower bounded by \( \frac{1}{5} \mathbb{E} \left[ \left( \frac{1}{2} - P_{ij} \right)^2 \right] \).

Under the condition 9, we conclude that (8) is satisfied.

### 7.3. Proof of Theorem 2

We prove the lemma using Fano’s inequality. By Stirling’s formula we have

\[ |Y| = \binom{n}{n/2} \geq 2^{n/2}. \]

Now suppose \( Y^* \) is sampled uniformly at random from \( \mathcal{Y} \), and then \( P \) and \( A \) are generated according to our model. We have

\[
I (A, P; Y^*) = H (A, P) - H (A, P | Y^*)
\leq \left( \frac{n}{2} \right) [H (A_{11}, P_{11}) - H (A_{11}, P_{11} | Y^*_{11})]
\leq n^2 I (A_{11}, P_{11} | Y^*_{11}),
\]

where the first inequality follows from symmetry. Let \( a = A_{11}, p = P_{11} \) and \( y = Y^*_{11} \). We now compute

\[
I (a, p; y) = \mathbb{E}_{a, p, y} \left[ \log \frac{\mathbb{P} (a, p | y)}{\mathbb{P} (a, p)} \right] = \mathbb{E}_{a, p, y} \left[ \log \frac{\mathbb{P} (a | y, p) \mathbb{P} (p)}{\mathbb{P} (a | p) \mathbb{P} (p)} \right]
= \mathbb{E}_{a, p, y} \left[ \log \mathbb{P} (a | y, p) \right] - \mathbb{E}_{a, p, y} \left[ \log \mathbb{P} (a | p) \right]
\leq \mathbb{E}_{a, p, y} \left[ \log \mathbb{P} (a | y, p) \right] - \mathbb{E}_{p} \left[ p \log \frac{1}{2} \right] + \mathbb{E}_{y} \left[ (1 - p) \log \frac{1}{2} \right].
\]
One verifies that
\[
E_{a,p,y} [\log P(a|y,p)] = E_{a,p,y} [E [\log P(a|y,p) | a, y]] = E_p [p \log p] + E_p [(1 - p) \log (1 - p)] .
\]

Combining the last three display equations gives
\[
I(A, P; Y^*) \leq n^2 E_p [p \log 2p + (1 - p) \log 2(1 - p)] \\
\leq n^2 E_p [p (2p - 1) + (1 - p) (1 - 2p)] \\
= n^2 E_p [(1 - 2p)^2] .
\]

It follows that under the condition (10) of the theorem, we have
\[
I(A, P; Y^*) \leq n^2 \cdot \frac{c'}{n} = c'n \leq \frac{1}{4} \log |\mathcal{Y}|
\]
provided \(c'\) is sufficiently small. Applying Fano’s inequality, we obtain that for any \(\hat{Y}\),
\[
P \left[ \hat{Y}(A, P) \neq Y^* \right] \geq 1 - \frac{I(A, P; Y^*) + \log 2}{\log |\mathcal{Y}|} \geq \frac{1}{2},
\]
where the probability is w.r.t. the randomness in \(Y^*, P\) and \(A\). Because the supremum is lower bounded by the average, we conclude that
\[
\sup_{Y^* \in \mathcal{Y}} P \left[ \hat{Y}(A, P) \neq Y^* \right] \geq \frac{1}{2}.
\]

Taking the infimum over all \(\hat{Y}\) proves the theorem.