Lemma S9. Let $\beta \in [0,1]$, $q \in [0,1]$ and $p \in (0,1)$. We have

$$D^*_\beta(q,p) = D_{KL}(q,p) + \frac{1-\beta}{\beta} D_{KL}\left(\frac{p-\beta q}{1-\beta}, p\right),$$

where $D_{KL}(\cdot, \cdot)$ and $D^*_\beta(\cdot, \cdot)$ are defined respectively by Equations (3) and (8) of the main paper.

Proof.

\[
D^*_\beta(q,p) = q \ln \frac{\beta q}{p} + \left(\frac{p}{\beta} - q\right) \ln \left(1 - \beta \frac{q}{p}\right) + \left(1 - q\right) \ln \frac{1 - q}{1 - p} + \left(\frac{1 - p}{\beta} + q - 1\right) \ln \left(1 - \frac{1 - q}{1 - p}\right) - \ln \beta - \left(\frac{1}{\beta} - 1\right) \ln (1 - \beta) \\
= D_{KL}(q,p) + \left(\frac{p}{\beta} - q\right) \ln \frac{1 - \beta \frac{q}{p}}{1 - \beta} + \left(\frac{1 - p}{\beta} + q - 1\right) \ln \frac{1}{1 - p} - \left(\frac{1}{\beta} - 1\right) \ln (1 - \beta) \\
= D_{KL}(q,p) + \left(\frac{p}{\beta} - q\right) \ln \frac{1 - \beta \frac{q}{p}}{1 - \beta} + \left(\frac{1 - p}{\beta} + q - 1\right) \ln \frac{1}{1 - p} - \left(\frac{1 - p}{\beta} + q - 1\right) \ln (1 - \beta) \\
= D_{KL}(q,p) + \left(\frac{p}{\beta} - q\right) \left[\ln \frac{1 - \beta \frac{q}{p} - 1}{\beta} - 1\right] + \left(\frac{1 - p}{\beta} + q - 1\right) \left[\ln 1 - \beta \frac{1 - q}{1 - p} - 1\right] \\
= D_{KL}(q,p) + \left(\frac{p}{\beta} - q\right) \ln \frac{1 - \beta \frac{q}{p}}{1 - \beta} + \left(\frac{1 - p}{\beta} + q - 1\right) \ln \frac{1 - \beta \frac{1 - q}{1 - p}}{1 - \beta} \\
= D_{KL}(q,p) \left[\frac{p - \beta q}{1 - \beta} \ln \frac{1 - \beta \frac{q}{p}}{1 - \beta} + \left(1 - \frac{p - \beta q}{1 - \beta}\right) \ln \frac{1 - \beta \frac{1 - q}{1 - p}}{1 - \beta}\right] \\
= D_{KL}(q,p) \left[\frac{p - \beta q}{1 - \beta} \ln \frac{1 - \beta \frac{q}{p}}{1 - \beta} + \left(1 - \frac{p - \beta q}{1 - \beta}\right) \ln \frac{1 - \beta \frac{1 - q}{1 - p}}{1 - \beta}\right].
\]
Lemma S10. Let $m, N, K$ be integers such that $\lambda \leq m \leq N - \lambda$ and $0 \leq K \leq N$. We have

$$F(k) = \frac{\alpha(k, K) \alpha(m-k, N-K)}{\alpha(m, N)} \leq e^{\frac{1}{12}} \sqrt{2\pi m(1 - \frac{m}{N})},$$

for $k = \max[0, K+m-N]$ and $k = \min[m, K]$.

Proof. First, let study the case $k = \max[0, K+m-N]$.

If $0 \geq K+m-N$, then $F(0) = \frac{\alpha(m,N-K)}{\alpha(m,N)}$ increases according to $K$, and its maximum is reached at $K = N-m$. We have

$$F(0) \leq \frac{\alpha(m, m)}{\alpha(m, N)} = \frac{1}{\alpha(m, N)}.$$

If $0 \leq K+m-N$, then $F(K+m-N) = \frac{\alpha(K+m-N, K) \alpha(N-K, N-K)}{\alpha(m, N)} = \frac{\alpha(K+m-N, K)}{\alpha(m, N)}$ decreases according to $K$, and its maximum is reached at $K = N-m$. Then

$$F(K+m-N) = F(0) \leq \frac{1}{\alpha(m, N)}.$$

Now, let us study the case $k = \min[m, K]$.

If $m \leq K$, then $F(m) = \frac{\alpha(m, K)}{\alpha(m, N)}$ decreases according to $K$, and its maximum is reached at $K = m$. We have

$$F(m) \leq \frac{\alpha(m, m)}{\alpha(m, N)} = \frac{1}{\alpha(m, N)}.$$

If $m \geq K$, then $F(K) = \frac{\alpha(K, K) \alpha(m-K, N-K)}{\alpha(m, N)} = \frac{\alpha(m-K, N-K)}{\alpha(m, N)}$ increases according to $K$, and its maximum is reached at $K = m$. Then

$$F(K) = F(m) \leq \frac{1}{\alpha(m, N)}.$$

Finally, by Lemma 3, we get

$$\frac{1}{\alpha(m, N)} \leq \frac{1}{\sqrt{\frac{N}{2\pi m(N-m)}} e^{-\frac{\frac{1}{12}}{m} e^{\frac{1}{12}} \frac{1}{2(N-m)}}} = \sqrt{\frac{2\pi m(1 - \frac{m}{N}) e^{\frac{1}{12}} \frac{1}{m} e^{\frac{1}{12}} \frac{1}{2(N-m)}}} \leq e^{\frac{1}{12}} \sqrt{2\pi m(1 - \frac{m}{N})}.$$
Lemma S11. Let \( m, N, K \) be integers such that \( 0 \leq m \leq N \) and \( 0 \leq K \leq N \). We have

\[
\sum_{k \in K_{mNK}^*} \sqrt{\left( \frac{1}{k} + \frac{1}{K-k} \right) \left( \frac{1}{m-k} + \frac{1}{(N-K)-(m-k)} \right)} \leq 2 \sum_{k=1}^{m-1} \frac{1}{k} \leq 2 (1 + \ln(m-1)),
\]

where

\[
K_{mNK}^* = \{ \max[0, K+m-N] + 1, \ldots, \min[m, K] - 1 \},
\]

and we have an equality at Line (20) when \( m = K = N - K \).

**Proof.** First, examine the case where \( m = K = N - K \).

\[
\sum_{k \in K_{mNK}^*} \sqrt{\left( \frac{1}{k} + \frac{1}{K-k} \right) \left( \frac{1}{m-k} + \frac{1}{(N-K)-(m-k)} \right)} = \sum_{k \in K_{mNK}^*} \sqrt{\left( \frac{1}{k} + \frac{1}{m-k} \right) \left( \frac{1}{m-k} + \frac{1}{N-K} \right)} = \sum_{k \in K_{mNK}^*} \sqrt{\left( \frac{1}{k} \right) \left( \frac{1}{m-k} + \frac{1}{N-K} \right)} = \sum_{k \in K_{mNK}^*} \left( \frac{1}{k} \right) + \sum_{k \in K_{mNK}^*} \frac{1}{m-k} = 2 \sum_{k=1}^{m-1} \frac{1}{k}.
\]

The last equality comes from the fact that when \( m = K = N - K \), the set \( K_{mNK}^* \) equals \( \{1, 2, \ldots, m-1\} \). The two sums are then equivalent.

Let us now examine all other cases. We distinguish 4 distinct cases, where each demonstration consists in using the case’s inequality such that the expression’s value raises.

**Case 1:** \( m \leq (N - K) \) and \( m \leq K \).

\[
\sum_{k \in K_{mNK}^*} \sqrt{\left( \frac{1}{k} + \frac{1}{K-k} \right) \left( \frac{1}{m-k} + \frac{1}{(N-K)-(m-k)} \right)} \leq \sum_{k \in K_{mNK}^*} \sqrt{\left( \frac{1}{k} + \frac{1}{m-k} \right) \left( \frac{1}{m-k} + \frac{1}{N-K} \right)} = \sum_{k \in K_{mNK}^*} \sqrt{\left( \frac{1}{k} \right) \left( \frac{1}{m-k} + \frac{1}{N-K} \right)} = \sum_{k=1}^{m-1} \frac{1}{k} + \sum_{k=1}^{m-1} \frac{1}{m-k} = 2 \sum_{k=1}^{m-1} \frac{1}{k}.
\]

**Case 2:** \( m \leq (N - K) \) and \( m > K \).

\[
\sum_{k \in K_{mNK}^*} \sqrt{\left( \frac{1}{k} + \frac{1}{K-k} \right) \left( \frac{1}{m-k} + \frac{1}{(N-K)-(m-k)} \right)} \leq \sum_{k \in K_{mNK}^*} \sqrt{\left( \frac{1}{k} + \frac{1}{K-k} \right) \left( \frac{1}{K-k} + \frac{1}{(N-K)-(m-k)} \right)} = \sum_{k \in K_{mNK}^*} \sqrt{\left( \frac{1}{k} \right) \left( \frac{1}{K-k} + \frac{1}{(N-K)-(m-k)} \right)} = \sum_{k=1}^{K-1} \frac{1}{k} + \sum_{k=1}^{K-1} \frac{1}{K-k} = 2 \sum_{k=1}^{K-1} \frac{1}{k} < 2 \sum_{k=1}^{m-1} \frac{1}{k}.
\]
Case 3: $m > (N - K)$ and $m \leq K$.

\[
\sum_{k \in K_{mNK}} \sqrt{\left(\frac{1}{k} + \frac{1}{K-k}\right) \left(\frac{1}{m-k} + \frac{1}{(N-K)-(m-k)}\right)} \\
\leq \sum_{k \in K_{mNK}} \sqrt{\left(\frac{1}{k} + \frac{1}{m-k}\right) \left(\frac{1}{m-k} + \frac{1}{(N-K)-(m-k)}\right)} \\
\leq \sum_{k \in K_{mNK}} \sqrt{\left(\frac{1}{(N-K)-(m-k)} + \frac{1}{m-k}\right) \left(\frac{1}{m-k} + \frac{1}{(N-K)-(m-k)}\right)} \\
= \sum_{k \in K_{mNK}} \left(\frac{1}{(N-K)-(m-k)} + \frac{1}{m-k}\right) = \sum_{k=m-N+K+1}^{m-1} \left(\frac{1}{(N-K)-(m-k)} + \frac{1}{m-k}\right) \\
= 2 \sum_{k=m-N+K+1}^{m-1} \frac{1}{m-k} < 2 \sum_{k=1}^{m-1} \frac{1}{k} = 2 \sum_{k=1}^{m-1} \frac{1}{k}.
\]

Case 4: $m > (N - K)$ and $m > K$.

\[
\sum_{k \in K_{mNK}} \sqrt{\left(\frac{1}{k} + \frac{1}{K-k}\right) \left(\frac{1}{m-k} + \frac{1}{(N-K)-(m-k)}\right)} \\
\leq \sum_{k \in K_{mNK}} \sqrt{\left(\frac{1}{k} + \frac{1}{K-k}\right) \left(\frac{1}{K-k} + \frac{1}{(N-K)-(m-k)}\right)} \\
\leq \sum_{k \in K_{mNK}} \sqrt{\left(\frac{1}{(N-K)-(m-k)} + \frac{1}{K-k}\right) \left(\frac{1}{K-k} + \frac{1}{(N-K)-(m-k)}\right)} \\
= \sum_{k \in K_{mNK}} \left(\frac{1}{(N-K)-(m-k)} + \frac{1}{K-k}\right) = \sum_{k=m-N+K+1}^{K-1} \left(\frac{1}{(N-K)-(m-k)} + \frac{1}{K-k}\right) \\
= 2 \sum_{k=m-N+K+1}^{K-1} \frac{1}{K-k} \leq 2 \sum_{k=1}^{K-1} \frac{1}{K-k} = 2 \sum_{k=1}^{m-1} \frac{1}{k} \leq 2 \sum_{k=1}^{m-1} \frac{1}{k}.
\]

For each case, we showed that the expression is lower or equal than $2 \sum_{k=1}^{m-1} \frac{1}{k}$. Using the approximation by definite integral technique, we obtain as needed

\[
2 \sum_{k=1}^{m-1} \frac{1}{k} \leq 2 \left(1 + \int_{1}^{m-1} \frac{1}{x} \, dx\right) = 2 \left(1 + \ln(m-1)\right).
\]

\[\square\]
**Theorem S12** (Fixed version of Derbeko et al. [2004], Theorem 18). For any set $Z$ of $N$ examples, for any set $\mathcal{H}$ of classifiers, for any prior distribution $P$ on $\mathcal{H}$, for any $\delta \in (0,1]$, with a probability at least $1-\delta$ over the choice $S$ of $m$ examples among $Z$,

$$\forall Q \text{ on } \mathcal{H}: \quad R_Z(G_Q) \leq R_S(G_Q) + \sqrt{\frac{1 - \frac{m}{N}}{2(m-1)}} \left[ \text{KL}(Q\|P) + \log \frac{m}{\delta} + 7 \log(N+1) \right].$$

**Proof.** Let us use the shortcut notations $R_S = R_S(G_Q)$ and $R_Z = R_Z(G_Q)$. We start from Equation (17) of Derbeko et al. [2004]:

$$D_{KL}(R_S, R_Z) + \frac{1 - \frac{m}{N}}{m} D_{KL} \left( \frac{R_Z - \frac{m}{N} R_S}{1 - \frac{m}{N}}, R_Z \right) - \frac{7}{m} \log(N+1) \leq \frac{\text{KL}(Q\|P) + \log \frac{m}{\delta}}{m-1}.$$

Applying Pinsker’s inequality ($D_{KL}(q,p) \geq 2(q-p)^2$) twice, we get

$$D_{KL}(R_S, R_Z) + \frac{1 - \frac{m}{N}}{m} D_{KL} \left( \frac{R_Z - \frac{m}{N} R_S}{1 - \frac{m}{N}}, R_Z \right) \geq 2(R_S - R_Z)^2 + 2(\frac{N}{m} - 1) \left( \frac{R_Z - \frac{m}{N} R_S}{1 - \frac{m}{N}} - R_Z \right)^2 = \frac{2(R_S - R_Z)^2}{1 - \frac{m}{N}}.$$

Hence, the result is obtained by isolating $R_Z$ in

$$\frac{2(R_S - R_Z)^2}{1 - \frac{m}{N}} - \frac{7}{m} \log(N+1) \leq \frac{\text{KL}(Q\|P) + \log \frac{m}{\delta}}{m-1}.$$

\[\square\]

**Remark.** Note that Derbeko et al. [2004] state their result as bound on $R_U(G_Q)$, i.e., a bound on the risk on the unlabeled examples. As

$$R_Z(h) = \frac{1}{N} \left( m R_S(h) + (N-m) R_U(h) \right),$$

the statement of Theorem S12 above can be directly converted from a bound on $R_Z(G_Q)$ to a bound of $R_U(G_Q)$. We then have

$$\frac{1}{N} \left( m R_S(h) + (N-m) R_U(h) \right) \leq R_S(G_Q) + \sqrt{\frac{1 - \frac{m}{N}}{2(m-1)}} \left[ \text{KL}(Q\|P) + \log \frac{m}{\delta} + 7 \log(N+1) \right],$$

and

$$R_U(h) \leq R_S(G_Q) + \sqrt{\frac{1}{2(m-1)(1 - \frac{m}{N})}} \left[ \text{KL}(Q\|P) + \log \frac{m}{\delta} + 7 \log(N+1) \right].$$
More Empirical Study of different $D$-functions

We show results similar to Figure 1, this time considering $R_S(G_Q) = 0.1$ and $R_S(G_Q) = 0.01$ in Figures 2 and 3. As these figures are generated in exactly the same fashion than Figure 1, we omit unnecessary explanation.

Figure 2: Study of the behavior of bounds obtained by Theorem 5. All graphics consider $R_S(G_Q) = 0.1$, $KL(Q\|P) = 5$ and $\delta = 0.05$. 
Figure 3: Study of the behavior of bounds obtained by Theorem 5. All graphics consider $R_S(G_Q) = 0.01$, $\text{KL}(Q\|P) = 5$ and $\delta = 0.05$. 