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Multivariate Spearman's ρ for Aggregating Ranks Using Copulas

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Abstract

We study the problem of rank aggregation: given a set of ranked lists, we want to form a consensus ranking. Furthermore, we consider the case of extreme lists: i.e., only the rank of the best or worst elements are known. We impute missing ranks and generalise Spearman's ρ to extreme ranks. Our main contribution is the derivation of a non-parametric estimator for rank aggregation based on multivariate extensions of Spearman's ρ, which measures correlation between a set of ranked lists. Multivariate Spearman's ρ is defined using copulas, and we show that the geometric mean of normalised ranks maximises multivariate correlation. Motivated by this, we propose a weighted geometric mean approach for learning to rank which has a closed form least squares solution. When only the best (top-k) or worst (bottom-k) elements of a ranked list are known, we impute the missing ranks by the average value, allowing us to apply Spearman's ρ. We discuss an optimistic and pessimistic imputation of missing values, which respectively maximise and minimise correlation, and show its effect on aggregating university rankings. Finally, we demonstrate good performance on the rank aggregation benchmarks MQ2007 and MQ2008.

1. Introduction

Ranking is a central task in many applications such as information retrieval, recommender systems and bioinformatics. It may also be a subtask of other learning problems such as feature selection, where features are scored according to their predictiveness, and then the most significant ones are selected. One major advantage of ranks over scores is that the resulting predicted ranks are automatically normalised and hence can be used to combine diverse sources of information. However, unlike many other supervised learning problems, the problem of learning to rank (Lebanon and Mao, 2008; Liu, 2011) does not have the simple one example one label paradigm. This has led to many formulations of learning tasks, depending on what label information is available, including pairwise ranking, listwise ranking and rank aggregation.

This paper considers a novel formulation of rank aggregation based on multivariate extensions to Spearman's ρ. For a set of n objects from the domain Ω, we are given a set of d experts that rank these objects providing rankings $R_1, \dots, R_d$. Each rank is a permutation of the n objects, and can be represented as a vector of unique integers from 1 to n. The problem of rank aggregation is to construct a new vector R that is most similar to the set of d ranks provided by the experts. In this paper we use Spearman's correlation ρ, a widely used correlation measure for ranks Spearman (1904). Instead of decomposing the association into a combination of pairwise similarities, $\rho(R, R_1), \rho(R, R_2), \dots, \rho(R, R_d)$, we directly maximise the multivariate correlation

$$R^* = \arg\max_R \rho(R, R_1, R_2, \dots, R_d).$$

Measures of association such as Spearman's ρ capture the concordance between random variables (Nelsen, 2006). Informally, random variables are concordant if large values of one tend to be associated with large values of the other. Let (x_i, y_i) and (x_j, y_j) be two observations of a pair of continuous random variables. We say that (x_i, y_i) and (x_j, y_j) are *concordant* if $x_i < x_j$ and $y_i < y_j$ or if $x_i > x_j$ and $y_i > y_j$. If the inequalities disagree, we say that the samples are *discordant*. The concept of concordance captures only the order of the random variables, and is invariant to their values, and therefore is ideal for analysing ranks. As will be described in section 3.3, Spearman's ρ is based on the difference between the concordance and discordance of the samples.

In short, Spearman's correlation can be defined as the concordance Q between the copula C corresponding to the data and the independent copula π

$$\rho \propto Q(C, \pi).$$

We review the concept of copulas in section 2 and derive our generalisation of concordance in section 3. While the mathematical machinery to derive our proposed algorithm relies on constructions that may not be familiar to some machine learners, the resulting algorithm for rank aggregation is straightforward. We solve a least squares problem for n items,

$$\min_\omega \sum_{x=1}^{n} \left(l(x) - \sum_{j=1}^{d} \omega_d r_d(x) \right)^2,$$

where we minimise the weights $\omega_1, \dots, \omega_d$ corresponding to the d experts. It turns out that the appropriate transformation to learn weights between experts is to use logarithmic scaled ranks. In the above equation, $l(x)$ and $r(x)$ denote the logarithm of the labels and individual expert ranks respectively, with all ranks normalised uniformly to the interval $(0, 1)$. Since it is a least squares problem, there is a closed form solution for the optimal weights. This is in contrast to previous approaches to rank aggregation that involve complex optimisation methods or sampling.

1.1 Our Contributions

We theoretically justify why the above least squares problem provides a meaningful way to weight experts. We show that the geometric mean of a set of normalised ranks maximises multivariate Spearman's ρ. This motivates our method which finds a setting of weights that maximise multivariate Spearman's ρ for a specific target (supervised rank aggregation).

As previously mentioned, in many applications of rank aggregation, only extreme ranks are available, whereas the standard definitions of Spearman's ρ require full ranks. For practical problems, the expert may only rank the most liked (top-k) or most disliked (bottom-k) objects where k can be different for each expert. We propose a method for estimating Spearman's ρ for extreme ranks by imputing the remaining ranks. We describe this method and show that it is an unbiased estimator in section 4.

This results in a non-parametric approach for rank aggregation that learns the weights of experts by solving a least squares problem. The weights in this case model dependencies between the rankings, i.e., the rankings are not independent. This is different to much prior work (see section 1.3) in that we explicitly learn the dependencies between experts simultaneously and not in a pairwise fashion. Our method thus offers significant computational benefits, modelling flexibility in the presence of dependencies between experts, and also interpretability due to the simplicity of the model. In section 6 we describe our empirical results for rank aggregation and show that our simple algorithm performs better than current state of the art results.

1.2 Multiple Representations of Ranks

There are a wide range of applications which benefit from rank analysis, resulting in various equivalent ways to represent ranks and orderings. The basic representation often used in introductory texts is to provide the list of objects, for example $[a, b, c, d, e, f]$, denoting the fact that a is the most highly ranked object and f is the lowest ranked. It is often more convenient to numerically represent the rank for computational purposes, that is to keep a list of integers $1, \ldots, n$ corresponding to the rank of a particular object. For the example above, by maintaining the set of objects as is, the ranks are then $[1, 2, 3, 4, 5, 6]$. It turns out for empirical copula modeling, it is important that the numerical values are in the interval $(0, 1)$, and therefore we normalise the numerical representation by $n + 1$, that is $[\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}]$. However, note that the numerical representation is actually dependent on the fact that we have maintained the set of objects in a particular fashion. In fact, by the above numerical list, we are saying that object a has rank $\frac{1}{7}$, and object f has rank $\frac{6}{7}$. In other words, we are defining a permutation mapping $R : \Omega \to (0, 1)$ from the space of objects Ω to the interval $(0, 1)$.

1.3 Related Work

There are two related rank aggregation tasks: score based rank aggregation and order based rank aggregation. For score based rank aggregation objects are associated with scores, while for order based rank aggregation only the relative order of objects are available. There has been recent work on combining both scores and ranks (Sculley, 2010; Iyer and Bilmes, 2013). We consider the learning task referred to as the listwise approach in Liu (2011), where the input is a set of ranked lists of documents from multiple experts, and the learner has to predict the final ranks. Numerous proposals for solving the problem of combining multiple lists into a single list are surveyed in Liu (2011). Niu et al. (2012) has focused on learning a good ranking from given features. A good review of probability and statistics applied to permutations is Diaconis (1988).

Spearman's ρ is a natural measure of similarity for distributions of permutations (Mallows, 1957; Fligner and Verducci, 1986). Interestingly, there has not been much work using Spearman's ρ for dealing with ranked data, but instead the focus has been on Kendall's τ. One difficulty of inference with the Mallows model (Mallows, 1957) for Spearman's ρ is that it involves estimating the permanent of a matrix. Our model is derived from the copula form of Spearman's ρ and allows a simple formulation for aggregation that does not require any computationally complex operations, thus providing a significant computational advantage.

Other previous approaches (Klementiev et al., 2008; Iyer and Bilmes, 2012) to rank aggregation considers pairwise comparisons between ranked lists. In contrast, our approach does not consider pairwise combinations and operates over all lists. We prove a result saying that the geometric mean of normalised ranks maximise Spearman's ρ (theorem 17), which is similar in spirit to the result in Iyer and Bilmes (2012) that shows that for Lovász–Bregman divergences the best aggregator is the arithmetic mean. This provides a computational advantage over pairwise methods as the number of lists grows.

Our work builds heavily on copula theory, and we use results from Nelsen (2006). Brief introductions to copulas can be found in Trivedi and Zimmer (2005), Genest and Favre (2007), and Elidan (2013). Further details on copula modeling are available in a recent book (Joe, 2014). Many of these results are presented for bivariate copulas only. There are fewer results on multivariate copulas (Joe, 1990; Nelsen, 1996) and their relation to Spearman's ρ (Úbeda Flores, 2005; Schmid et al., 2010), which we shall discuss later in this paper.

Finally, other well known measures of bivariate dependence have forms under the copula framework and have multivariate extensions. In particular, multivariate extensions of Kendall's τ have been proposed (Joe, 2014). It is possible investigations into these copula formulations results in other efficient aggregation methods with different tradeoffs, however in this work we focus on Spearman's ρ.

The work on partial ranks goes back to at least Critchlow (1985), who describes the rank aggregation task in terms of distances between rankings. We have applied the results of this paper to rank aggregation (Macintyre et al., 2014) and stability estimation (Bedő et al., 2014) in the domain of life sciences.

2. Copulas

Copulas are functions from the unit hypercube to the unit interval (Elidan, 2013). In this section we briefly review the bivariate setting, in preparation for the multivariate setting in the next section. The expert reader may skip directly to section 3 to see the definition of multivariate Spearman's ρ in terms of the multivariate copula.

2.1 Definition of Copulas

Intuitively, for continuous random variables copulas model the dependence component of a multivariate distribution after discounting for univariate marginal effects. We let $\mathbb{R}$ denote the ordinary real line $(-\infty, \infty)$, and $\overline{\mathbb{R}}$ denote the extended real line $[-\infty, \infty]$. The following algebraic definition of bivariate copulas is generalised to the multivariate setting in section 3.

It essentially constrains copulas to be functions that are *monotonically increasing* along each dimension as well as towards the diagonal of the volume.

Definition 1 *Let A_1 and A_2 be nonempty subsets of $\overline{\mathbb{R}}$, and let $H(\cdot,\cdot)$ be a real function such that the domain of $H = A_1 \times A_2$. Let $B = [x_1, x_2] \times [y_1, y_2]$ be a rectangle all of whose vertices are in the domain of H. Then the* H-volume *of B is given by:*

$$V_H(B) = H(x_1, y_1) + H(x_2, y_2) - H(x_1, y_2) - H(x_2, y_1).$$

Definition 2 *A real function $H(\cdot,\cdot)$ is* 2-increasing *if its H-volume is non-negative, that is $V_H(B) \geqslant 0$ for all rectangles B whose vertices lie in the domain of H.*

Definition 3 *A* copula *is a function $C\colon [0,1]^2 \to [0,1]$ with the following properties:*

1. *For every $u, v \in [0,1]$,*
$$C(u,0) = 0 = C(0,v)$$
$$C(u,1) = u \quad \text{and} \quad C(1,v) = v$$
2. *C is 2-increasing.*

2.2 Relation Between Bivariate Cumulative Density Functions and Copulas

Sklar's theorem is central to the theory of copulas and is the foundation of many applications in statistics. Indeed, Sklar's theorem can be defined for general distribution functions outside of probabilistic settings. However, since we are interested in statistical applications we will consider cumulative distribution functions.

Theorem 4 (Sklar's theorem) *Let $H(\cdot,\cdot)$ be a cumulative distribution function with marginals $F(\cdot)$ and $G(\cdot)$. Then there exists a copula $C\colon [0,1]^2 \to [0,1]$ such that for all x, y in $\overline{\mathbb{R}}$,*

$$H(x,y) = C(F(x), G(y)).$$

If $F(\cdot)$ and $G(\cdot)$ are continuous then $C(\cdot,\cdot)$ is unique; otherwise $C(\cdot,\cdot)$ is uniquely determined on the ranges of $F(\cdot)$ and $G(\cdot)$.

Conversely, if $C(\cdot,\cdot)$ is a copula and $F(\cdot)$ and $G(\cdot)$ are cumulative distribution functions then the function $H(\cdot,\cdot)$ is a bivariate cumulative distribution function with marginals $F(\cdot)$ and $G(\cdot)$.

3. Spearman's ρ

We briefly review the bivariate model to lay out the approach for estimating the copula using data, the so-called empirical copula.

3.1 Empirical bivariate Spearman's ρ

Let R and S be ranking functions, which are bijections mapping elements x in the domain U to $[1, 2, \ldots, n]$. The domain U represents the space of objects that we are interested in ranking, such as documents retrieved in response to a query or the biomarkers most associated with a disease. Since we consider only the ranks of the object $R(x)$ and $S(x)$, the actual domain U does not affect the analysis. The sums below are over the n objects x. Similar to the approach of Pearson's correlation for the measure of dependence, Spearman's ρ is a measure of correlation between ranks, empirically given by:

$$\rho_n = \frac{\sum_x (R(x) - \bar{R})(S(x) - \bar{S})}{\sqrt{\sum_x \left(R(x) - \bar{R}\right)^2 \sum_x \left(S(x) - \bar{S}\right)^2}}, \tag{1}$$

where $\bar{R} := \frac{1}{n}\sum_x R(x)$ and $\bar{S} := \frac{1}{n}\sum_x S(x)$ are the empirical means of the respective random variables. This is equivalent to applying Pearson's correlation to the ranks instead of the values of the score function itself. There is no direct way to generalise this expression to more than two ranking functions, but as we shall see in section 3.3 we can obtain an expression via the copula.

By substituting the definitions of the empirical means and rearranging the terms, we obtain

$$\rho_n = \left(\frac{n+1}{n-1}\right)\left[\frac{12}{n}\sum_x \frac{R(x)}{n+1}\frac{S(x)}{n+1} - 3\right].$$

The constants 12 and 3 seem strange, but are a natural consequence of the mean and variance of a list of ranks. As we will see later, these constants are dependent only on the dimension of the copula. Similar to the definition of an empirical CDF, we define an empirical copula as:

$$C_n(u,v) = \frac{1}{n}\sum_x \mathbf{1}\left(\frac{R(x)}{n+1} \leqslant u, \frac{S(x)}{n+1} \leqslant v\right),$$

where $\mathbf{1}$ is the indicator function. This allows us to re-express the form of ρ_n above in terms of an integral over the unit square,

$$\rho_n = \left(\frac{n+1}{n-1}\right)\left[\frac{12}{n}\sum_x \frac{R(x)}{n+1}\frac{S(x)}{n+1} - 3\right] = \left(\frac{n+1}{n-1}\right)\left[12\int_{[0,1]^2} uvC_n(u,v) - 3\right].$$

It can be shown (Nelsen, 2006; Genest and Favre, 2007) that ρ_n is an asymptotically unbiased estimator of

$$\rho = 12\int_{[0,1]^2} C(u,v)\,\mathrm{d}u\,\mathrm{d}v - 3,$$

where C is the population version of C_n.

3.2 Multivariate Copulas

We now generalise the definitions in section 2.1 to the multivariate case. The concepts are essentially the same, constraining the copula to be "monotonically increasing" in the interval $[0,1]$ and also towards the center of the volume (Durante and Sempi, 2010).

Definition 5 *Let A_j be nonempty subsets of $\overline{\mathbb{R}}$ for $j = 1,\ldots,d$, and let $H_d\colon A_1 \times \cdots \times A_d \to \mathbb{R}$. Let $B = [a_1,b_1] \times \cdots \times [a_d,b_d]$ be the d-box where all vertices are contained in* Dom H_d. *Then the H_d-*volume *of B is the d^{th} order difference:*

$$V_{H_d}(B) = \Delta_{a_d}^{b_d} \ldots \Delta_{a_1}^{b_1} H_d(\vec{t}),$$

where

$$\Delta_{a_i}^{b_i} H(\vec{t}) = H_d(t_1, \ldots, t_{i-1}, b_i, t_{i+1}, \ldots, t_d) - H_d(t_1, \ldots, t_{i-1}, a_i, t_{i+1}, \ldots, t_d).$$

Definition 6 *A real function H_d is* grounded *if $H_d(\vec{t}) = 0$ for all $t \in \operatorname{Dom} H_d$ such that $t_j = a_j$ for at least one $j \in \{1, \ldots, d\}$.*

Definition 7 *A real function H_d is* d-increasing *if $V_{H_d}(B) \geqslant 0$ for all n-boxes B whose vertices lie in the domain of H.*

Definition 8 *A multivariate* copula *has the following properties:*

1. $\operatorname{Dom} C = [0,1]^d$
2. *C has margins $C_j(u) = C(1, \ldots, 1, u, 1, \ldots, 1) = u$ for all j and $u \in I$*
3. *C is grounded*
4. *C is d-increasing.*

There is an alternative probabilistic definition that may be more familiar to readers with a statistical background.

Definition 9 *Let $U_1, \ldots, U_d$ be real uniformly distributed random variables on the unit interval $\sim U([0,1])$. A copula function $C\colon [0,1]^d \longrightarrow [0,1]$ is a joint distribution*

$$C(u_1, \ldots, u_d) = P(U_1 \leqslant u_1, \ldots, U_d \leqslant u_d).$$

Let $X \sim F$ be a continuous random variable such that the inverse of the CDF F^{-1} exists. What is the distribution of $F(x) = P(X \leqslant x)$?

$$\begin{aligned} P(F(X) \leqslant u) &= P(F^{-1}(F(X)) \leqslant F^{-1}(u)) \\ &= P(X \leqslant F^{-1}(u)) \\ &= F(F^{-1}(u)) = u \end{aligned}$$

The above calculation shows that the distribution is uniform, i.e. $F(x) \sim U([0,1])$. This can be considered to be the *copula trick*, as the user has the freedom to choose the copula independently of the marginal distributions.

3.3 Multivariate Extension of Spearman's ρ

We generalise the concept of concordance to the multivariate setting such that we can define multivariate Spearman's ρ in an analogous way to the bivariate ρ as defined in Nelsen (2006).

Recall that two random variables are concordant if they tend to be in the same order, that is (x_i, y_i) and (x_j, y_j) are concordant if $(x_i - x_j)(y_i - y_j) > 0$, and are discordant if $(x_i - x_j)(y_i - y_j) < 0$. The concordance function Q denotes the difference between the probabilities of concordance and discordance, and as the following theorem shows, can be expressed in terms of the copulas. The proof is in Nelsen (2006).

Theorem 10 (Concordance function) *Let* (X_1, Y_1) *and* (X_2, Y_2) *be two independent vectors with joint distributions* $H_1(x,y) = C_1(F(x), G(y))$ *and* $H_2(x,y) = C_2(F(x), G(y))$ *respectively. Then the concordance function* Q *is given by*

$$\begin{aligned} Q(C_1, C_2) :=& P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0] \\ =& 4 \int_{[0,1]^2} C_2(u,v) \, \mathrm{d}C_1(u,v) - 1 \end{aligned}$$

We now state the generalisation of concordance to the multivariate case (Nelsen, 1996; Joe, 1990). Further details of multivariate concordance can be found in Taylor (2007) and Schmid et al. (2010).

Definition 11 (Multivariate concordance) *Let* $(X_1, \dots, X_d)$ *and* $(Y_1, \dots, Y_d)$ *be two independent d-vectors with joint distributions* $C_X(F(x))$ *and* $C_Y(F(y))$ *where* $F(x) = F_1(x_1), \dots, F_d(x_d)$ *and* $F(y) = F_1(y_1), \dots, F_d(y_d)$ *are the marginal distributions, and* C_X, C_Y *are the respective d copulas. Then the concordance function* Q *is given by*

$$Q(C_X, C_Y) := 2^d \int_{[0,1]^d} C_X(u) \, \mathrm{d}C_Y(u) - 1.$$

Note that although the integral is a straight forward generalisation of theorem 10, it is no-longer equal to the difference between the probability of concordance and discordance. Consequently, the properties possessed by Q are different for $d > 2$.

There are three copulas that are of particular interest: the independent copula $\pi(u) := \prod_i u_i$, and the upper and lower Fréchet–Hoeffding bounds, $M(u) = \min\{u_1, u_2, \dots, u_d\}$ and $W(u) \geq \max\{u_1 + u_2 + \cdots + u_d - (d-1), 0\}$ respectively (Joe, 2014, pg. 48). Note that that while W is point-wise sharp, this lower bound is not itself a copula, and hence the lower bound is not tight (Úbeda Flores, 2005).

Theorem 12 *Let* C, C', *and* Q *be given as in definition 11,* M *and* W *be the upper and lower Fréchet–Hoeffding bounds respectively, and assume* $d > 2$. *Then*

1. Q *is symmetric in its arguments if* $C = C'$.
2. Q *is non-decreasing in the first argument, and both arguments if* $C = C'$.
3. $-1 \leq Q(W,W) \leq Q(C,C) \leq Q(M,M) = 2^{d-1} - 1$.
4. $Q(\pi, \pi) = 0$.

Proof

Property 1 The first property is clear from the definition of $Q(C, C')$ and the properties of integration.

Property 2 Q is non-decreasing in the first argument by properties of integration. For the second part, notice that

$$\begin{aligned} \int C(u) \, \mathrm{d}C(u) =& C^2(u) - \int C(u) \, \mathrm{d}C(u) \\ \Rightarrow \int C(u) \, \mathrm{d}C(u) =& \frac{1}{2} C^2(u) \end{aligned}$$

by applying integration by parts. The property now follows.

Property 3 It follows that

$$\begin{aligned} Q(M,M) =& 2^d \int_{[0,1]^d} M(u)\, \mathrm{d}M(u) - 1 \\ =& 2^d \int_0^1 u\, \mathrm{d}u - 1 \\ =& 2^{d-1} - 1, \end{aligned}$$

and

$$\begin{aligned} Q(W,W) =& 2^d \int_{[0,1]^d} W(u)\, \mathrm{d}W(u) - 1 \\ \geq& 2^d \int_0^1 0\, \mathrm{d}u - 1 \\ =& -1. \end{aligned}$$

Property 3 now follows from the first two properties.

Property 4

$$\begin{aligned} Q(\pi,\pi) =& 2^d \int_{[0,1]^d} \pi(u)\, \mathrm{d}\pi(u) - 1 \\ =& 2^d \int_{[0,1]^d} u\, \mathrm{d}u - 1 \\ =& \frac{2^d}{2^d} - 1 \\ =& 0 \end{aligned}$$

■

It is clear from this theorem that Q is well calibrated at $Q(W,W)$ and $Q(\pi,\pi)$, however not for $Q(M,M)$. Consequently, with this multidimensional extension it becomes increasingly difficult to estimate discordance as d increases.

Proposition 13 *Let Q be given as in definition 11, and M and π be the upper Fréchet–Hoeffding bound and the independent copula respectively, then*

$$Q(M,\pi) = Q(\pi,M) = \frac{2^d - (d+1)}{d+1}. \tag{2}$$

Proof To show the symmetry,

$$\begin{aligned} Q(M,\pi) =& 2^d \int_{[0,1]^d} M(u)\, \mathrm{d}\pi(u) - 1 \\ =& 2^d \int_{[0,1]^d} u_1 u_2 \cdots u_d\, \mathrm{d}u - 1 \end{aligned}$$

and

$$\begin{aligned} Q(\pi, M) =& 2^d \int_{[0,1]^d} \pi(u)\, \mathrm{d}M(u) - 1 \\ =& 2^d \int_{[0,1]^d} u_1 u_2 \cdots u_d\, \mathrm{d}u - 1. \end{aligned}$$

To obtain the second equality, we observe that

$$\begin{aligned} \int_{[0,1]^d} u_1 u_2 \cdots u_d\, \mathrm{d}u =& \int_0^1 u^d\, \mathrm{d}u \\ =& \frac{1}{d+1} u^{d+1} \Big|_0^1 \\ =& \frac{1}{d+1}, \end{aligned}$$

and therefore the expression for $Q(M, \pi)$ follows. ■

In terms of the concordance function, Spearman's ρ is given by the concordance between the copula C and the independent copula $\pi(u) := \prod_i u_i$. However, unlike the symmetry in proposition 13, the concordance function is in general not symmetric with respect to its arguments. This gives us two possible ways of defining multivariate Spearman's ρ, corresponding to $Q(C, \pi)$ and $Q(\pi, C)$. Both generalisations are equivalent in the bivariate case, and has been called ρ_d^- and ρ_d^+ by Nelsen (1996) and ρ_1 and ρ_2 by Schmid and Schmidt (2007) respectively. Naturally, there is a third symmetric generalisation which is the average of them.

Definition 14 (Multivariate Spearman's ρ)

$$\rho_d^- = h(d) Q(\pi, C) = h(d) \left[2^d \int_{[0,1]^d} C(u)\, \mathrm{d}u - 1 \right] \tag{3}$$

and

$$\rho_d^+ = h(d) Q(C, \pi) = h(d) \left[2^d \int_{[0,1]^d} \pi(u)\, \mathrm{d}C(u) - 1 \right], \tag{4}$$

where $h(d) = \frac{d+1}{2^d - (d+1)}$ *is the normalisation factor.*

The scaling factor $h(d)$ is derived such that the maximum correlation is 1. Thus, for Spearman's ρ, this is the concordance between the maximum copula M and the independent copula π, which we obtain by proposition 13:

$$h(d) = 1/Q(M, \pi) = \frac{d+1}{2^d - (d+1)}. \tag{5}$$

Spearman's correlation can equivalently be seen as measuring average orthant dependence, and the two versions ρ_d^+ and ρ_d^- correspond to whether we look at the upper or lower orthant (Nelsen, 1996). Positive upper orthant dependence is defined as

$$P(X > x) \geq \prod_{i=1}^{d} P(X_i > x_i),$$

and positive lower orthant dependence is defined as

$$P(X \leq x) \geq \prod_{i=1}^{d} P(X_i \leq x_i).$$

When $d = 2$, the two definitions are the same and are called positive quadrant dependence (Lehmann, 1966), as we have already observed for the concordance function:

$$\begin{aligned} P(X_1 > x_1, X_2 > x_2) &\geq P(X_1 > x_1)P(X_2 > x_2) \\ &\geq [1 - P(X_1 \leq x_1)][1 - P(X_2 \leq x_2)] \\ &\geq 1 - P(X_1 \leq x_1) - P(X_2 \leq x_2) + P(X_1 \leq x_1)P(X_2 \leq x_2). \end{aligned}$$

Rearranging gives

$$P(X_1 > x_1, X_2 > x_2) + P(X_1 \leq x_1) + P(X_2 \leq x_2) - 1 \geq P(X_1 \leq x_1)P(X_2 \leq x_2).$$

The left hand side is $P(X_1 \leq x_1, X_2 \leq x_2)$.

Observe that the scaling factor $h(d)$ is the same for both ρ_d^- and ρ_d^+ due to proposition 13. Furthermore, since $P(X_i > x_i) = 1 - P(X_i \leq x_i)$ for each random variable, the two versions of Spearman's ρ correspond to looking at whether we interpret the ranks as top down or bottom up. Converting from one version to the other can be done by reinterpreting the data. For a particular application, the choice of which version to use depends on the ranks that are available. We will focus on ρ_d^+ henceforth.

Recall that for a set of n objects from the domain Ω, we are given a set of d experts that rank these objects providing ranks $R_1, \ldots, R_d$, where each R_j is a bijection to $(0, 1)$. Putting (4) and (5) together, we obtain the following expression for multivariate Spearman's correlation:

$$\rho(R_1, \ldots, R_d) = h(d)Q(C, \pi) = \frac{d+1}{2^d - (d+1)} \left[2^d \int_{[0,1]^d} \pi(u) \, \mathrm{d}C(u) - 1 \right]. \tag{6}$$

In practice, we do not have access to the population version of the copula $C(u)$ but have the empirical copula $C_n(u)$. We discuss this further in section 5.

Unlike the bivariate case, as the number of dimensions increases, the lower bound of Spearman's ρ tends to zero. This counterintuitive fact can be understood by considering the three dimensional case. Consider three rankings R_1, R_2, and R_3. If R_1 and R_2 are anti-correlated (ρ=-1), and at the same time R_1 and R_3 are also anti-correlated, this implies that R_2 and R_3 must be perfectly correlated (ρ=1). Hence, the overall 3 dimensional correlation is no longer -1. This can be made precise by considering the inclusion-exclusion principle, which results in the following relation from Nelsen (1996):

$$\frac{1}{2}(\rho_d^-(R_1,R_2,R_3)+\rho_d^+(R_1,R_2,R_3))=\frac{1}{3}(\rho(R_1,R_2)+\rho(R_1,R_3)+\rho(R_2,R_3)).$$

The following corollary defines the lower bound as the number of dimensions increases.

Corollary 15 *Under the minimum Fréchet–Hoeffding bound* W*,* $Q(W,\pi)\geq -1$ *and*

$$\lim_{d\to\infty}\rho(R_1,\ldots,R_d)\geqslant h(d)Q(W,\pi)=0.$$

In particular, for dimension d*,*

$$\rho(R_1,\ldots,R_d)\geqslant\frac{2^n-(n+1)!}{n!(2^n-(n+1))}.$$

Proof This follows immediately from the bound $-1\leq Q(W,\pi)\leq 0$ (from theorem 12) since $h(d)$ goes to zero as $d\to\infty$. The lower bound has also been observed in Nelsen (1996) and Schmid et al. (2010). ∎

In summary, the multivariate extension of Spearman's correlation is still calibrated under maximum correlation as it achieves a value of 1, but it becomes increasingly difficult to observe anti-correlated sets of ranks as the number of lists to be aggregated increases. In the next section, we investigate an aggregation algorithm that maximises correlation. The effect of the lower bound is discussed with respect to imputing missing values in section 5.

4. Optimal Aggregation with Spearman's ρ

The empirical copula requires R and S to comprise of ranks for the same set of elements, that is $\operatorname{Dom} R=\operatorname{Dom} S$. Recall from section 3.1 that ranks map to the range $\{1,\ldots,n\}$, but the empirical copula is expressed in terms of fractional ranks (divided by $n+1$). In the following it is convenient to work with normalised ranks, that is to consider R and S as bijections to $(0,1)$. The expression for the empirical copula then simplifies to

$$C_n(u,v)=\frac{1}{|\Omega|}\sum_{x\in\Omega}\mathbf{1}\left(R(x)\leqslant u,S(x)\leqslant v\right),\tag{7}$$

where is the domain of the objects we are interested in ranking. Correspondingly, the d dimensional empirical copula for n objects given by

$$C_n(u)=\frac{1}{n}\sum_{x}\prod_{j=1}^{d}\mathbf{1}\left(R_j(x)\leqslant u_j\right),\tag{8}$$

where $R_1(x),\ldots,R_d(x)$ is the rankings of the d experts. Plugging the empirical copula (8) expression into Spearman's ρ (6), and observing that integrating the product over the copula is the product of the ranks Schmid and Schmidt (2007), we obtain an empirical expression for multivariate Spearman's correlation:

$$\rho_n(R_1,\ldots,R_d)=h(d)\left[\frac{2^d}{n}\sum_{x}\prod_{j=1}^{d}R_j(x)-1\right].\tag{9}$$

4.1 Geometric Mean is Optimal

We are now in a position to derive the deceptively simple result: the ranking R that maximises correlation with a given set of rankings $\{R_1, \ldots, R_d\}$ is given by the geometric mean of $R_1, \ldots, R_d$. The following definition is needed to capture the notion that ranks only depend on the order.

Definition 16 (Rank generator) $\sigma\colon \mathbb{R}^{|\Omega|} \to [0,1]^{|\Omega|}$ *is a* rank generator *if:*

- *for all $x, y \in \Omega$ and R with domain Ω, $R(x) < R(y) \iff \sigma \circ R(x) < \sigma \circ R(y)$;*
- *for any rankings R, R' with domain Ω there exists a permutation ξ such that $\sigma \circ R' = \sigma \circ \xi \circ R$;*
- *for any permutation ξ, $\xi \circ \sigma = \sigma \circ \xi$.*

A rank generator formalises the idea of generating a rank: the ranks it generates must be invariant to scale and only dependent on the ordering of elements. The standard ranking functions from statistics such as fractional ranking and dense ranking fit into this framework.

Theorem 17 *Let $\{R_1, R_2, \ldots, R_d\}$ be a set of rankings with common domain Ω and σ be a rank generator. Then*

$$\arg \max_{R \in \operatorname{codom} \sigma} \rho_n(R, R_1, R_2, \ldots, R_d) = \sigma\left(\prod_{j=1}^{d} R_j\right).$$

Proof Consider the expression for Spearman's ρ_n (9):

$$\rho_n(R, R_1, R_2, \ldots, R_d) = h(d+1)\left[\frac{2^{d+1}}{n}\sum_{x}\left(R(x)\prod_{j=1}^{d} R_j(x)\right) - 1\right].$$

Focusing on the terms in the sum, showing that the best possible $R(x)$ is $\prod_{j=1}^{d} R_j(x)$ reduces to showing

$$\sum_{x \in U} \sigma \circ P(x) P(x)$$

is maximal, where $P := \prod_j R_j$. Suppose there exists an P' such that

$$\sum_{x \in U} \sigma \circ P'(x) P(x) > \sum_{x \in U} \sigma \circ P(x) P(x).$$

By definition of σ, there exists a permutation ξ such that

$$\begin{aligned}
\sum_{x \in U} \sigma \circ P'(x) P(x) &= \sum_{x \in U} \sigma \circ \xi \circ P(x) P(x) \\
&= \sum_{x \in U} \xi \circ \sigma \circ P(x) P(x) \\
&> \sum_{x \in U} \sigma \circ P(x) P(x).
\end{aligned}$$

This is a contradiction for any permutation ξ as σ is order preserving. ■

Corollary 18 *The converse applies, that is:*

$$\arg\min_{R \in \operatorname{codom} \sigma} \rho_n(R, R_1, R_2, \ldots, R_d) = \sigma\left(\prod_{j=1}^{d}(1 - R_j)\right).$$

Proof Proof follows from a similar argument. ■

5. Empirical Copulas with Partial Lists

In many applications it is prohibitive to obtain complete annotations of the object ranks. For example, in the document retrieval setting, this amounts to providing ranks for all documents. The empirical copula requires the set of rankings $\{R_1, \ldots, R_d\}$ to comprise of ranks for the same set of elements, that is $\operatorname{Dom} R_1 = \cdots = \operatorname{Dom} R_d$. Hence, a key challenge in applying Spearman's ρ to rank aggregation is to estimate the statistic on incompletely labelled lists.

Recall the definition of the empirical copula (7). We now consider the case where $\operatorname{Dom} R \neq \operatorname{Dom} S$, but R and S are generated from two *top ranked* lists. We define extended rankings R', S' with codomain $[0, 1]$ such that $\operatorname{Dom} R' = \operatorname{Dom} R \cup \operatorname{Dom} S = \operatorname{Dom} S'$. One way to impute the missing values is to set them to a constant value for all the ranks below the top-k ranks. This value is chosen to be the mid point between the start and end of the missing section. The values in the top-k are retained to be the original values in the extension. The definition below formally defines this notion. Note that we have to renormalise the values.

Definition 19 (non-informative extension) *Let R be a ranking operator and R' be its extension to domain* $\operatorname{Dom} R'$. *Then,*

$$R'(x) = \begin{cases} \frac{|\operatorname{Dom} R|}{|\operatorname{Dom} R'|} R(x) & x \in \operatorname{Dom} R \\ \frac{|\operatorname{Dom} R| + |\operatorname{Dom} R'|}{2|\operatorname{Dom} R'|} & \text{otherwise} \end{cases} \tag{10}$$

$\forall x \in \operatorname{Dom} R'$.

We call this the non-informative extension since it assumes that all items that are not ranked have the same rank (the mean of the missing ranks). Note that the two experts R_i and R_j may have ranked different numbers of objects. An advantage of this extension is that it can easily deal with the case of more than two experts. Consider d experts $R_1, \ldots, R_d$, each of which may have ranked a different subset of the objects. Hence the extension has to impute values on the union of items from all experts. Denote $\operatorname{Dom} R' := \operatorname{Dom} R_1 \cup \ldots \cup \operatorname{Dom} R_d$, then we can apply definition 19 to complete each ranking operator R_j. An additional advantage to the non-informative extension is that it results in a consistent ranking.

Definition 20 *An extended ranking R' of R is called* consistent *if the following axioms hold:*

1. $R'(x) < R'(y)$ $\forall x, y \in \operatorname{Dom} R$ *with* $R(x) < R(y)$
2. $R'(x) = R'(y)$ $\forall x, y \in \operatorname{Dom} R$ *with* $R(x) = R(y)$
3. $R'(y) > R'(x)$ $\forall x \in \operatorname{Dom} R$, $y \in \operatorname{Dom} R'$

If $E[R] = E[R']$ also holds, then R' is called strictly consistent*.*

Lemma 21 *Definition 19 produces a consistent ranking. If $E[R] = \frac{1}{2}$ then* (10) *produces a strictly consistent ranking.*

Proof The notation $|\operatorname{Dom} R|$ can become unwieldly in following proof. We therefore adopt the shorthand notations $r := |\operatorname{Dom} R|$ and $r' := |\operatorname{Dom} R'|$ for the size of the respective sets.

Axioms 1 and 2 are satisfied by definition as the map $x \mapsto \frac{r}{r'}x$ is monotonic. For all $x \in \operatorname{Dom} R' \setminus \operatorname{Dom} R$,

$$R'(x) = \frac{r + r'}{2r'} \leqslant 2R(y)\frac{r + r'}{2r'} \leqslant \frac{2R(y)r}{2r'} = \frac{r}{r'}R(y) = R'(y)$$

for any $y \in \operatorname{Dom} R$, satisfying axiom 3.

Furthermore, as

$$\begin{aligned}
E[R'] &= \frac{1}{r'}\left(\sum_{x \in \operatorname{Dom} R} R'(x) + \sum_{x \in \operatorname{Dom} R' \setminus \operatorname{Dom} R} R'(x)\right) \\
&= \frac{1}{r'}\left(\frac{r}{r'}\sum_{x \in \operatorname{Dom} R} R(x) + (r' - r)\frac{r + r'}{2r'}\right) \\
&= \frac{1}{r'}\left(\frac{r^2}{r'}E[R] + (r' - r)\frac{r + r'}{2r'}\right) \\
&= \frac{r^2(2E[R] - 1) + r'}{2r'^2},
\end{aligned}$$

R' is strictly consistent if $E[R] = \frac{1}{2}$. ■

Definition 19 is called a *non-informative* extension as it uses no additional information and does not bias the imputed elements in anyway: imputed values are all considered tied and mapped to the same value. Furthermore, the strictly consistent property that definition 19 satisfied is important when using fractional ranking as it guarantees no introduction of bias.

Note also that there is a dual imputation whereby missing values are assigned to the top of the list rather than the bottom. This is equivalent to the above imputation applied to reverse rankings. The choice of top or bottom imputation is application dependent.

5.1 Empirical Upper and Lower Bounds

Proposition 22 *For top-k lists where k of n items are ranked by all d experts with codomain* $\{1,\ldots,n\}$ *(i.e., unnormalised ranks), the Spearman's* ρ *is bounded by*

$$\rho_n(R_1,R_2,\ldots,R_d)=\rho_k(R_1,R_2,\ldots,R_d)+C,$$

where

$$\frac{2^d\,\mathrm{h}\,(d)\,\left(\left(k\,(k+1)^d-n\,(n+1)^d\right)\,\left(\sum_{i=1}^k\prod_{j=1}^d\frac{R_j(i)}{k+1}\right)+k\,\sum_{i=k+1}^n i^{\frac{d}{2}}\,(k-i+n+1)^{\frac{d}{2}}\right)}{n\,(n+1)^d\,k}$$
$$\le C\le$$
$$\frac{2^d\,\mathrm{h}\,(d)\,\left(\left(k\,(k+1)^d-n\,(n+1)^d\right)\,\left(\sum_{i=1}^k\prod_{j=1}^d\frac{R_j(i)}{k+1}\right)+\left(\sum_{i=k+1}^n i^d\right)\,k\right)}{n\,(n+1)^d\,k}.$$

Proof Proof sketch: the definition of ρ for unnormalised rankings is

$$\rho_n(R_1,R_2,\ldots,R_d)=h(d+1)\left[\frac{2^{d+1}}{n}\sum_{i=1}^n\left(\prod_{j=1}^d\frac{R_j(i)}{n+1}\right)-1.\right]$$

By considering the difference $\rho_n(R_1,R_2,\ldots,R_d)-\rho_k(R_1,R_2,\ldots,R_d)$ and factorising out the common terms, we obtain

$$C=\frac{(d+1)\,2^d\,\left(k\,\left(\sum_{i=1}^n\prod_{j=1}^d\frac{R_j(i)}{n+1}\right)-\left(\sum_{i=1}^k\prod_{j=1}^d\frac{R_j(i)}{k+1}\right)\,n\right)}{(2^d-d-1)\,k\,n}.$$

The term $\sum_{i=1}^n\prod_{j=1}^d\frac{R_j(i)}{n+1}$ can be bounded above by

$$\sum_{i=1}^n\prod_{j=1}^d\frac{R_j\,(i)}{n+1}=\sum_{i=1}^k\prod_{j=1}^d\frac{R_j\,(i)}{n+1}+\sum_{i=1+k}^n\prod_{j=1}^d\frac{R_j\,(i)}{n+1}\le\sum_{i=1}^k\prod_{j=1}^d\frac{R_j\,(i)}{n+1}+\sum_{i=1+k}^n\left(\frac{i}{n+1}\right)^d,$$

and below by

$$\begin{aligned}\sum_{i=1}^n\prod_{j=1}^d\frac{R_j\,(i)}{n+1}&=\sum_{i=1}^k\prod_{j=1}^d\frac{R_j\,(i)}{n+1}+\sum_{i=1+k}^n\prod_{j=1}^d\frac{R_j\,(i)}{n+1}\\&\ge\sum_{i=1}^k\prod_{j=1}^d\frac{R_j\,(i)}{n+1}+\sum_{i=k+1}^n\left(\prod_{j=i}^{\lceil\frac{d}{2}\rceil}\frac{i}{n+1}\right)\prod_{j=\lceil\frac{d}{2}\rceil}^d\frac{n-i+1+k}{n+1},\end{aligned}$$

giving us the bounds in the proposition. ∎

5.2 Optimal Imputation

An alternative to the previously presented imputation method is to impute such that ρ is maximised or minimised. In general this is a NP-hard problem as it involves searching all permutations. In this section, we formulate this as an optimisation problem.

Let $\mathbb{I} = \{1, \ldots, n\} \times \{1, \ldots, d\}$ be indices over n items and d experts. Let $\mathbb{O} \subset \mathbb{I}$ be the observed indices (for which we have a rank) and define $\mathbb{U} := \mathbb{I} \setminus \mathbb{O}$. We then have a rank function $R \colon \mathbb{O} \to \{1, \ldots, n\}$. Recall that Spearman's ρ is determined by a sum of the products over ranks. By introducing a log transformation, we convert the product into a sum using the logarithm rule:

$$\sum_{i=1}^{n} \left(\prod_{j=1}^{d} \frac{R_j(i)}{n+1} \right) = \sum_{i=1}^{n} \left(\exp \log \prod_{j=1}^{d} \frac{R_j(i)}{n+1} \right) = \sum_{i=1}^{n} \left(\exp \sum_{j=1}^{d} \log \frac{R_j(i)}{n+1} \right).$$

5.2.1 Imputing to Maximise Correlation

We can maximise Spearman's ρ by introducing binary indicators $x_{i,j,k}$ indexed over $\mathbb{I} \times \{1, \ldots, n\}$ to denote a rank of k for item i in list j.

$$\max_{x_{i,j,k}} \sum_{i=1}^{n} \exp \left[\sum_{j=1}^{d} \sum_{k=1}^{n} x_{i,j,k} \log \left(\frac{k}{n+1} \right) \right]$$

such that

$$\sum_{k} x_{i,j,k} = 1 \qquad \forall i, j \qquad (11)$$

$$\sum_{i} x_{i,j,k} = 1 \qquad \forall k, j \qquad (12)$$

$$\sum_{k} x_{i,j,k} k = R(i,j) \qquad \forall (i,j) \in \mathbb{O} \qquad (13)$$

$$x_{i,j,k} \in \{0, 1\} \qquad \forall i, j, k \qquad (14)$$

Constraint (11) ensures an item is only assigned one rank per expert, and constraint (12) ensures a rank is only assigned once per expert. Finally, the third constraint (13) ensures known ranks are assigned.

5.2.2 Imputing to Minimise Correlation

Analogously, we can consider the problem of minimising Spearman's ρ.

$$\min_{x_{i,j,k}} \sum_{i=1}^{n} \exp \left[\sum_{j=1}^{d} \sum_{k=1}^{n} x_{i,j,k} \log \left(\frac{k}{n+1} \right) \right] \qquad (15)$$

such that

$$\sum_k x_{i,j,k} = 1 \qquad \forall i, j$$

$$\sum_i x_{i,j,k} = 1 \qquad \forall k, j$$

$$\sum_k x_{i,j,k} k = R(i,j) \qquad \forall (i,j) \in \mathbb{O}$$

$$x_{i,j,k} \in \{0,1\} \qquad \forall i, j, k$$

By considering the relaxation of $x_{i,j,k} \in \{0,1\}$ to $x_{i,j,k} \in [0,1]$, we obtain a convex optimisation problem.

Proposition 23 *The relaxation of optimisation problem* (15) *such that* $x_{i,j,k}$ *is in the interval* $[0,1]$ *is a convex optimisation problem.*

Proof The objective has the form $\sum_i \exp(\langle x_i, \omega \rangle)$ with $\omega \in [\log\left(\frac{1}{n+1}\right), \ldots, \log\left(\frac{n}{n+1}\right)]^{nd}$. Thus, as each term in the sum is convex, and as the sum of convex functions is convex, the objective is convex. The constraints are all linear, hence this is a convex optimisation problem. ■

However, as a consequence of corollary 15, as $d \to \infty$ we know that $\rho \geq 0$, hence the minimum ρ will approach the ρ when using the non-informative extension (the non-informative extension has $\rho = 0$), thus there is little need to solve the optimisation problem after a sufficient number of dimensions is reached.

5.3 Experiments on University Ranking

The optimal imputation algorithm presented in section 5.2 is difficult to solve due to the integer constraints. We evaluated the performance of a relaxed version of the program, whereby the constraints are relaxed such that the variables may take a value in the range $[0,1]$. To solve the relaxed problem, we used a BFGS based optimiser by shifting the equality constraints into the objective function with high penalties. Final ranks were determined by ranking item i in list j based on the score $\sum_k x_{i,j,k} k$.

We evaluated this relaxed solution on imputing rankings for universities. To this end, the top-200 universities ranked by QS in 2014, Shanghai in 2014, and Times in 2015 were obtained. In aggregating these three lists, there are a total of 266 ranks that need to be imputed.

Measuring multivariate Spearman's ρ on all three lists imputing the missing elements using the non-informative extension gives $\rho = 0.632$. In comparison, the relaxed optimal imputation found a solution that obtained $\rho = 0.683$, a modest increase in the correlation. We also developed an interactive website[1] showing the detailed results for all universities, which also allows the user to alter the weights of each of the original experts. The top 36 aggregate rankings for the universities are given in appendix B.

1. `http://uni.cua0.org`

6. Supervised Learning to Rank

We now consider the task of learning rank aggregation from extreme ranks. Theorem 17 and definition 19 provide the core of our algorithm. Using theorem 17, we can find an average rank that aggregates a set of ranks, and by extending top-k and bottom-k ranks to a common domain, we can apply it to partially labelled data.

6.1 Weighted Mixture of Experts

As a result of theorem 17 we have a way of finding the ranking (according to some rank generator) that is closest to a set of ranks. Consider the learning problem where we have a ranking L which comprise our labels, and a set of d experts $\{R_j\}$. During training, we would like to find a weighting of the input rankings such that it gives the label. Given a target ranking L, we would like to optimise the weights ω,

$$\max_{\omega} \rho_n(L, R_1^{\omega_1}, R_2^{\omega_1}, \ldots, R_n^{\omega_d}).$$

Here we have introduced weights ω over each rank to control the influence of each rank over the final consensus rank; the intuition here is that ranks with $\omega_i > 1$ are replicated with more influence, which is easy to see when ω_i are natural numbers. For example, a weight of 2 would mean the ranked list has appeared twice in the calculation of the consensus rank. While it is convenient to have integer weights for interpretability, the weights ω could be any real number in general. In the following, we consider $\omega \in \mathbb{R}^n$. Instead of performing this high-dimensional optimisation, we decompose it into a pairwise (bivariate) comparison between the label L and the weighted geometric mean, where we now explicitly show the fact that the ranks are a function of the n objects x

$$\max_{\omega} \sum_x \rho_n(L(x),\, \sigma(R_1^{\omega_1} \otimes R_2^{\omega_2} \otimes \cdots \otimes R_d^{\omega_n})(x)),$$

where the notation $\otimes$ indicates the product operator. Observe that we have used theorem 17 to convert the d dimensional problem into the product of ranks R_j and the Spearman's correlation above is only two dimensional. For bivariate Spearman's ρ, this can be expressed in terms of the squared difference (1). We further assume that σ is the identity mapping to simplify the problem, giving us:

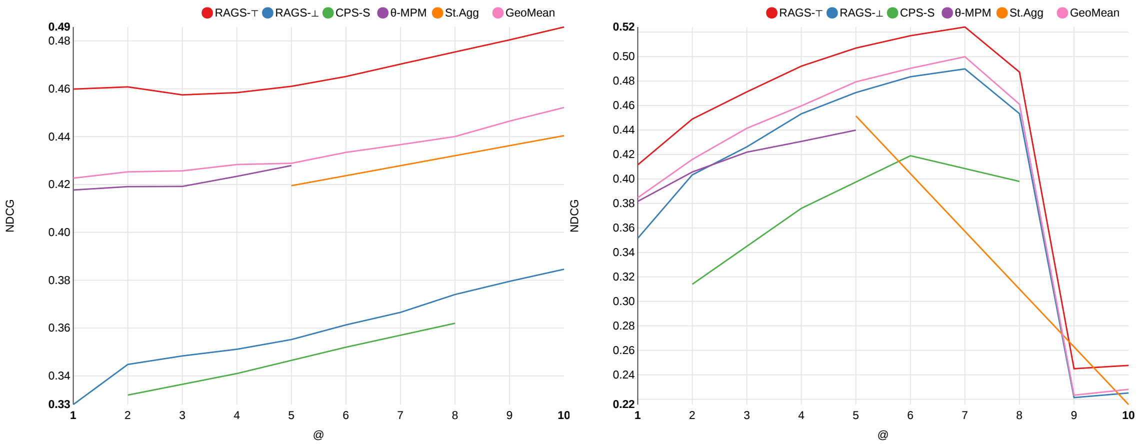
$$\min_{\omega} \sum_x \left(L(x) - R_1^{\omega_1}(x) R_2^{\omega_2}(x) \ldots R_d^{\omega_n}(x)\right)^2. \tag{16}$$

The objective (16) minimises the distance between the label ranks and the weighted expert ranks.

6.2 Least Squares Method on Logarithm of Ranks

Recall that we consider normalised ranks (divided by $n+1$). By using the logarithm identity, we convert the power scaling in (16) into a multiplicative scaling. Our algorithm is:

1. Extend incomplete ranks $\{R_i\}$ to $\{R_i'\}$ by imputing the average missing value such that $\operatorname{Dom} R_i' = \operatorname{Dom} L$;



(a) MQ2007 (b) MQ2008

Figure 1: Results on MQ2007-agg (a, left) and MQ2008-agg (b, right): NDCG@k. Our method is labelled RAGS- and RAGS- corresponding to top and bottom non-informative imputation respectively. The results for CPS-S was the best reported in Qin et al. (2010a). The results of θ-MPM was the best among the reported results in Volkovs and Zemel (2012) from BordaCount, CPS, SVP, Bradley-Terry model, and Plackett-Luce model. The results of St.Agg was the best among the reported results in Niu et al. (2013) and was the best among MCLK, SVP, Plackett-Luce model, θ-MPM, BordaCount and RRF.

2. Convert to log-ranks $r_i' = \log \circ R_i'$ and $l = \log \circ L$;

3. Learn weights ω by minimising

$$\sum_x \left(l(x) - \sum_{j=1}^{d} \omega_j r_j'(x) \right)^2, \qquad \text{where the outer sum is over the } n \text{ examples } x.$$

A log transformation of the ranks is used as it naturally encodes the weights as a power scaling in the framework of theorem 17, i.e., the weighted consensus rank is given by $\prod_i r_i'(x)^{\omega_i}$. Note that this is still solving (16) as we are optimising Spearman's ρ, which is sensitive only to ordering, and therefore though the final weights are different ρ is maximised via (1).

In the following experiments we also included a bias/offset term in the least squares problem, which can be interpreted as adding a ranking that is constant (gives all objects the same rank). It is interesting to note that the final step in this procedure is closely related to Borda Count, except our consensus rank is the geometric mean instead of the arithmetic mean. Since this is a least squares estimation problem, we directly use the closed form solution.

6.3 Benchmarking on LETOR 4.0

We tested our method on the MQ2007-agg and MQ2008-agg list aggregation benchmarks Qin et al. (2010b). The goal in these challenges is to aggregate 21 and 25 different rankers respectively over a set of query-document pairs. Each data set has 5 pre-defined cross-validation folds with each fold providing a training, testing and validation data set (60%/20%/20%). We trained our model on the training set and tested on the testing set, leaving the validation set unused since we have no hyperparameters.

In the following we consider two types of experts: either experts $\{R_j\}$ are top-k experts, that is they only rank the best k samples from Ω, or experts are bottom-k experts, that is they identify the worst k samples from Ω. We call our proposed method RAGS- and RAGS- respectively. We assume that the ranked documents in the benchmark data sets are either top-k or bottom-k respectively, with potentially different numbers of documents k labelled by each expert. Ties are given the average rank of tied documents.

To evaluate the agreement, we use the standard evaluation tool from the LETOR website[2], which implements the Normalised Discounted Cumulative Gain (NDCG). In fig. 1a, we see that our approach RAGS- performs better than all other methods at any selection size on the MQ2007-agg data set. Indeed, we also perform better than Qin et al. (2010a) where the best result uses a coset-permutation distance based stagewise (CPS) model with Spearman's ρ in a probabilistic model. Recall that our approach considers the multivariate Spearman's ρ whereas Qin et al. (2010a) uses bivariate Spearman's ρ in a pairwise fashion. For MQ2008-agg (fig. 1b), again our approach performs better than all other methods.

To tease apart the effect of imputing missing ranks and the effect of weighting the experts, we compared our proposed method with and without training (uniform weights). GeoMean denotes the results for the geometric mean (uniform weights on the experts) after performing imputation assuming top-k ranking by the experts. First we observe that our proposed approach outperforms the geometric mean, which is a good sanity check. It is surprising that the geometric mean performs quite well in MQ2007. The major difference is that we are imputing the missing ranks, and the other methods suffer from assigning them to an arbitrary value. This demonstrates the importance of imputation.

6.4 Strictly Ordered Labels

One issue with the benchmark aggregation data set is that the labels are only {0,1,2} relevance scores, and hence it is unclear exactly what the rankings are within the relevance classes. We create a new data set which is formed by taking the intersection between the documents retrieved by a particular query between MQ2007-agg and MQ2007-list. This new data set contains the strictly ordered labels from MQ2007-list, but uses the aggregation data from MQ2007-agg. The same procedure is used to create the corresponding data set for MQ2008-agg and MQ2008-list. These data sets are available for download at the LETOR website. We maintain exactly the same 5-fold cross validation splits and report our results in table 3.

Considering the results for Spearman's ρ, we observe that our learning method performs well. Note that the geometric mean outperforms Borda count on both data sets, which confirms that our theoretically justified model performs better than the heuristic model. It is interesting to observe that optimising for Spearman's ρ could result in a decrease in Kendall's τ. This demonstrates the importance of choosing the appropriate objective function for learning.

2. `http://research.microsoft.com/letor`

Table 1: Results on MQ2007-agg: NDCG. Our method is labelled RAGS- and RAGS- corresponding to top and bottom non informative imputation respectively. The results for CPS-S was the best reported in Qin et al. (2010a). The results of θ-MPM was the best among the reported results in Volkovs and Zemel (2012) from BordaCount, CPS, SVP, Bradley-Terry model, and Plackett-Luce model. The results of St.Agg was the best among the reported results in Niu et al. (2013) and was the best among MCLK, SVP, Plackett-Luce model, θ-MPM, BordaCount and RRF.

Fold	@1	@2	@3	@4	@5	@6	@7	@8	@9	@10
RAGS-	0.45986	0.46078	0.45744	0.45838	0.46102	0.46512	0.4703	0.47538	0.48042	0.4858
RAGS-	0.32804	0.3448	0.34836	0.35114	0.3552	0.36132	0.36656	0.37402	0.37952	0.38458
CPS-S		0.332		0.341		0.352		0.362		
-MPM	0.4177	0.4191	0.4192	0.4234	0.4279					
St.Agg					0.4195					0.4404
GeoMean	0.42264	0.42528	0.42570	0.42834	0.42886	0.43342	0.43664	0.44004	0.44648	0.45216

Table 2: Results on MQ2008-agg: NDCG

Fold	@1	@2	@3	@4	@5	@6	@7	@8	@9	@10
RAGS-	0.41158	0.44898	0.47118	0.4922	0.50696	0.51706	0.52416	0.48732	0.24498	0.24768
RAGS-	0.35156	0.40338	0.42624	0.45326	0.4706	0.48352	0.48994	0.45336	0.22138	0.22514
CPS-S		0.314		0.376		0.419		0.398		
-MPM	0.3817	0.4057	0.4219	0.4307	0.4399					
St.Agg					0.4515					0.2157
GeoMean	0.38470	0.41600	0.44142	0.45976	0.47938	0.49042	0.49986	0.46108	0.22334	0.22812

Table 3: Results on MQ2007-agglist and MQ2008-agglist. The left column shows the results for multivariate Spearman's ρ and the right column shows the result for Kendall's τ.

	MQ2007-agglist		MQ2008-agglist	
Method	ρ	τ	ρ	τ
RAGS-	0.4394	0.6201	0.7235	0.6931
RAGS-	0.2992	0.2488	0.6349	0.5560
GeoMean	0.2457	0.3011	0.5777	0.6578
Borda	0.2217	0.1790	0.5519	0.5869

7. Discussion and Conclusion

We propose an approach for learning weights between experts for the task of rank aggregation. By generalising the derivation of concordance functions, we obtain an expression for multivariate Spearman's ρ. Furthermore, we show that the geometric mean of the expert ranks is the optimal aggregator under Spearman's correlation. Motivated by this, our method solves a least squares estimation problem for logarithmic normalised ranks to find optimal weights.

One possible extension of our work is to compute the correlation for all possible subsets of rankings. While corollary 15 shows that the overall correlation cannot be negative as the number of rankings increase, there may be subgroups which are positively correlated within groups but negatively correlated between groups. By computing the correlation on the power set, we could use a clustering method to find such subgroups.

Though we have focused on ρ_d^+, our results are equally applicable to ρ_d^-; indeed it is a simple reversal of ranks that give ρ_d^-. The choice between ρ_d^+ and ρ_d^- is thus problem dependent: for tasks where being ranked highly is more informative ρ_d^+ is a better choice; conversely ρ_d^- is more suitable for tasks where being ranked lowly is more informative.

In contrast to other rank aggregation approaches, our method is very computationally efficient. However, the core of our method requires a complete set of rankings and hence does not handle missing variables. To resolve this, we propose three imputation methods (unbiased, optimistic, pessimistic) for completing top-k ranked lists that allows us to apply Spearman's ρ to aggregate ranks from partial lists. Our method is thus applicable for large scale applications with top-k rankings that arise in areas such as text mining and bioinformatics. One subtlety is that imputation from top-k should not be confused with the choice of using ρ_d^+, which is an separate design choice.

Surprisingly, our weighted geometric mean shows state of the art results on benchmark data sets, without the need for tuning hyperparameters or expensive computation. The simplicity of our model makes it easier to interpret, and the weights give a direct estimate of the influence of each expert. This problem has wide applications to ensemble learning, voting, text mining, recommender systems and bioinformatics.

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Appendix A. Bivariate Spearman's ρ and Squared Distance

This well known result[3] shows that Spearman's ρ can be expressed in terms of the squared distance between ranks.

In the following derivation, we use the expressions for the sum of integers and the sum of squares of integers:

$$\sum_{k=1}^{n} k_i = \frac{n(n+1)}{2} \qquad \sum_{k=1}^{n} k_i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Recall that Spearman's ρ is defined (1) as:

$$\rho_n = \frac{\sum_x (R(x)-\bar{R})(S(x)-\bar{S})}{\sqrt{\sum_x \left(R(x)-\bar{R}\right)^2 \sum_x \left(S(x)-\bar{S}\right)^2}}.$$

Since there are no ties, both $R(x)$ and $S(x)$ consist of integers from 1 to n inclusive, and the two squared sums in the denominator are the same. Recall that the mean rank is

$$\bar{R} = \bar{S} = \frac{n+1}{2}, \quad \text{and} \quad \sum_x R(x) = \frac{n(n+1)}{2} = n\bar{R}.$$

Therefore, the denominator can be expressed as a function of n:

$$\begin{aligned}
\sqrt{\sum_x \left(R(x)-\bar{R}\right)^2 \sum_x \left(S(x)-\bar{S}\right)^2} &= \sum_x \left(R(x)-\bar{R}\right)^2 \\
&= \sum_x (R(x)^2 - 2R(x)\bar{R} + \bar{R}^2) \\
&= \sum_x R(x)^2 - 2\bar{R}\sum_x R(x) + n\bar{R}^2 \\
&= \sum_x R(x)^2 - n\bar{R}^2 \\
&= \frac{n(n+1)(2n+1)}{6} - n\left(\frac{n+1}{2}\right)^2 \\
&= n(n+1)\left(\frac{2n+1}{6} - \frac{n+1}{4}\right) \\
&= n(n+1)\left(\frac{n-1}{12}\right) \\
&= \frac{n(n^2-1)}{12}.
\end{aligned}$$

3. `http://en.wikipedia.org/wiki/Spearman's_rank_correlation_coefficient` accessed on 20 May 2014

Since both $R(x)$ and $S(x)$ consists of the same integers, we can express the squared difference in terms of the product.

$$\begin{aligned}\sum_x \frac{1}{2}(R(x) - S(x))^2 &= \sum_x \frac{1}{2}(R(x)^2 - 2R(x)S(x) + S(x)^2) \\ &= \sum_x \frac{1}{2}(R(x)^2 + S(x)^2) - \sum_x R(x)S(x) \\ &= \sum_x R(x)^2 - \sum_x R(x)S(x),\end{aligned}$$

where the first term is a function of n.

We express the product of the means, which appears in the numerator later, to match the denominator:

$$\begin{aligned}n\left(\frac{n+1}{2}\right)^2 &= \frac{n(n+1)}{12}3(n+1) \\ &= -\frac{n(n+1)}{12}[(n-1) - (4n+2)] \\ &= -\frac{n(n+1)(n-1)}{12} + \frac{n(n+1)(2n+1)}{6} \\ &= -\frac{n(n^2-1)}{12} + \sum_x R(x)^2,\end{aligned}$$

where the last term in the sum is the expression for the sum of squares. We can now derive the expression for the numerator:

$$\begin{aligned}\sum_x (R(x) - \bar{R})(S(x) - \bar{S}) &= \sum_x R(x)S(x) - \bar{R}\sum_x S(x) - \bar{S}\sum_x R(x) + n\bar{R}\bar{S} \\ &= \sum_x R(x)S(x) - n\bar{R}\bar{S} \\ &= \sum_x R(x)S(x) - n\left(\frac{n+1}{2}\right)^2 \\ &= \sum_x R(x)S(x) + \frac{n(n^2-1)}{12} - \sum_x R(x)^2 \\ &= \frac{n(n^2-1)}{12} - \sum_x \frac{1}{2}(R(x) - S(x))^2,\end{aligned}$$

where the last line uses the expression of the sum of squared differences above. Putting together the expressions for the numerator and denominator together gives the desired result:

$$\rho_n = 1 - \frac{6\sum_x (R(x) - S(x))^2}{n(n^2-1)}.$$

Appendix B. University Aggregate Rankings

Rank	University
1	Harvard University
2	Massachusetts Institute of Technology
3	Stanford University
4	California Institute of Technology
5	University of Cambridge
6	University of Oxford
7	Princeton University
8	University of Chicago
9	University of California, Berkeley
10	Imperial College London
11	ETH Zurich
12	University College London
13	Yale University
14	Columbia University
15	Johns Hopkins University
16	Cornell University
17	University of California, Los Angeles
18	University of Pennsylvania
19	University of Michigan
20	University of Toronto
21	Duke University
22	Northwestern University
23	University of Edinburgh
24	University of California, San Diego
25	King's College London
26	University of Washington
27	University of Tokyo
28	National University of Singapore
29	New York University
30	École Polytechnique Fédérale de Lausanne
31	McGill University
32	University of Melbourne
33	University of Illinois at Urbana–Champaign
34	University of Wisconsin–Madison
35	University of British Columbia
36	University of Manchester
37	Australian National University

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