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Convergence Rate of Optimal Quantization and Application to the Clustering Performance of the Empirical Measure

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Abstract

We study the convergence rate of the optimal quantization for a probability measure sequence $(\mu_n)_{n\in\mathbb{N}^*}$ on $\mathbb{R}^d$ converging in the Wasserstein distance in two aspects: the first one is the convergence rate of optimal quantizer $x^{(n)} \in (\mathbb{R}^d)^K$ of μ_n at level K; the other one is the convergence rate of the distortion function valued at $x^{(n)}$, called the "performance" of $x^{(n)}$. Moreover, we also study the mean performance of the optimal quantization for the empirical measure of a distribution μ with finite second moment but possibly unbounded support. As an application, we show an upper bound with a convergence rate $\mathcal{O}(\frac{\log n}{\sqrt{n}})$ of the mean performance for the empirical measure of the multidimensional normal distribution $\mathcal{N}(m, \Sigma)$ and of distributions with hyper-exponential tails. This extends the results from Biau et al. (2008) obtained for compactly supported distribution. We also derive an upper bound which is sharper in the quantization level K but suboptimal in n by applying results in Fournier and Guillin (2015).

Keywords: clustering performance, convergence rate of optimal quantization, distortion function, empirical measure, optimal quantization

1. Introduction

The K-means clustering procedure in the unsupervised learning area was first introduced by MacQueen (1967), which consists in partitioning a data set of observations $\{\eta_1, ..., \eta_N\} \subset \mathbb{R}^d$ into K classes $\mathcal{G}_k, 1 \le k \le K$ with respect to a *cluster center* $x = (x_1, ..., x_K)$ in order to minimize the quadratic distortion function $\mathcal{D}_{K,\eta}$ defined by

$$x = (x_1, ..., x_K) \in (\mathbb{R}^d)^K \mapsto \mathcal{D}_{K,\eta}(x) \coloneqq \frac{1}{N}\sum_{n=1}^{N} \min_{k=1,...,K} d(\eta_n, x_k)^2, \tag{1}$$

where d denotes a distance on $\mathbb{R}^d$. The classification of the observations $\{\eta_1, ..., \eta_N\} \subset \mathbb{R}^d$ in MacQueen (1967) can be described as follows

$$\mathcal{G}_1 = \Big\{ \ \eta_n \in \{\eta_1, ..., \eta_N\} \ : \ d(\eta_n, x_1) \le \min_{2\le j\le K} d(\eta_n, x_j) \ \Big\},$$

$$\mathcal{G}_2 = \big\{\ \eta_n \in \{\eta_1, ..., \eta_N\}\ :\ d(\eta_n, x_2) \leq \min_{1\leq j\leq K, j\neq 2} d(\eta_n, x_j)\ \big\} \setminus \mathcal{G}_1,$$

$$\cdots$$

$$\mathcal{G}_K = \big\{\ \eta_n \in \{\eta_1, ..., \eta_N\}\ :\ d(\eta_n, x_K) \leq \min_{1\leq j\leq K-1} d(\eta_n, x_j)\ \big\} \setminus \big(\mathcal{G}_{K-1} \cup \cdots \cup \mathcal{G}_1\big). \tag{2}$$

If a cluster center $x^* = (x_1^*, ..., x_K^*)$ satisfies $\mathcal{D}_{K,\eta}(x^*) = \inf_{y\in(\mathbb{R}^d)^K} \mathcal{D}_{K,\eta}(y)$, we call x^* an optimal cluster center (or *K-means*) for the observation $\eta = (\eta_1, ..., \eta_N)$. Such an optimal cluster center always exists but is generally not unique.

K-means clustering has a close connection with quadratic optimal quantization, originally developed as a discretization method for the signal transmission and compression by the Bell laboratories in the 1950s (see IEEE Transactions on Information Theory (1982) and Gersho and Gray (2012)). Nowadays, optimal quantization has also become an efficient tool in numerical probability, used to provide a discrete representation of a probability distribution. To be more precise, let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^d$ induced by the canonical inner product $\langle\cdot|\cdot\rangle$ and let X be an $\mathbb{R}^d$-valued random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with probability distribution μ having a finite second moment. The quantization method consists in discretely approximating μ by using a K-tuple $x = (x_1, ..., x_K) \in (\mathbb{R}^d)^K$ and its weight $w = (w_1, ..., w_K)$ as follows,

$$\mu \simeq \widehat{\mu}^x \coloneqq \sum_{k=1}^{K} w_k \delta_{x_k},$$

where δ_a denotes the Dirac mass at a, the weights w_k are computed by $w_k = \mu\big(C_k(x)\big), k = 1, ..., K$, and $\big(C_k(x)\big)_{1\leq k\leq K}$ is a *Voronoï partition* induced by x, that is, a Borel partition on $\mathbb{R}^d$ satisfying

$$C_k(x) \subset V_k(x) \coloneqq \big\{\xi \in \mathbb{R}^d\ \big|\ |\xi - x_k| = \min_{1\leq j\leq K} |\xi - x_j|\ \big\},\ \ k = 1, ..., K.$$

The value K in the above description is called the *quantization level* and the K-tuple above $x = (x_1, ..., x_K)$ is called a *quantizer* (or *quantization grid, codebook* in the literature). Moreover, we define the *(quadratic) quantization error function* $e_{K,\mu}$ of μ (or *of* X) at level K by

$$x = (x_1, ..., x_K) \in (\mathbb{R}^d)^K \longmapsto e_{K,\mu}(x) \coloneqq \Big[\int_{\mathbb{R}^d} \min_{1\leq k\leq K} |\xi - x|^2 \mu(d\xi)\Big]^{1/2}. \tag{3}$$

The set $\operatorname{argmin} e_{K,\mu}$ is not empty (see e.g. Graf and Luschgy, 2000, Theorem 4.12) and any element $x^* = (x_1^*, ..., x_K^*)$ in $\operatorname{argmin} e_{K,\mu}$ is called a *(quadratic) optimal quantizer* for the probability distribution μ at level K. Moreover, we call

$$e^*_{K,\mu} = \inf_{y=(y_1,...,y_K)\in(\mathbb{R}^d)^K} e_{K,\mu}(y) \tag{4}$$

the *optimal (quadratic) quantization error* (*optimal error* for short) at level K.

The connection between K-means clustering and quadratic optimal quantization is the following: if the distance d in (1) and (2) is the Euclidean distance and if we consider the empirical measure $\bar{\mu}_N$ of the data set $\{\eta_1, ..., \eta_N\}$ defined by

$$\bar{\mu}_N \coloneqq \frac{1}{N} \sum_{n=1}^{N} \delta_{\eta_n},$$

then the distortion function $\mathcal{D}_{K,\eta}$ defined in (1) is in fact $e^2_{K,\bar{\mu}_N}$ and $\operatorname{argmin} \mathcal{D}_{K,\eta} = \operatorname{argmin} e_{K,\bar{\mu}_N}$. That is, an optimal quantizer of $\bar{\mu}_N$ is in fact an optimal cluster center for the data set $\{\eta_1, ..., \eta_N\}$.

In Figure 1, we show an optimal quantizer and its weights for the standard normal distribution $\mathcal{N}\big(\mathbf{0}, \mathbf{I}_2\big)$ in $\mathbb{R}^2$ at level 60, where $\mathbf{I}_d$ denotes the identity matrix of size $d \times d$. The color of the cells in the figure represents the weight of each point x_k in the quantizer $x = (x_1, ..., x_K)$. In Figure 2, we show an optimal cluster center at level $K = 20$ for an i.i.d simulated sample $\{\eta_1, ..., \eta_{500}\}$ of the $\mathcal{N}(0, \mathbf{I}_2)$ distribution.

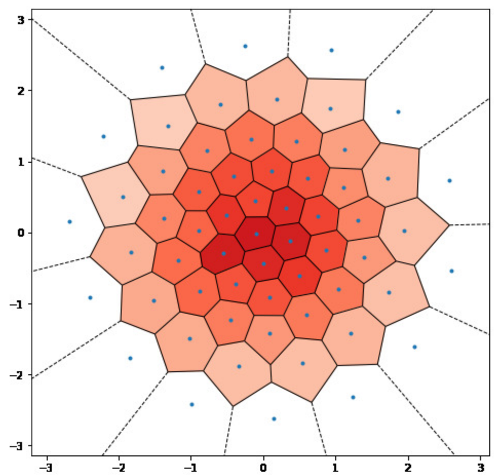

Figure 1: An optimal quantizer for $\mathcal{N}\big(\mathbf{0}, \mathbf{I}_2\big)$ at level 60.

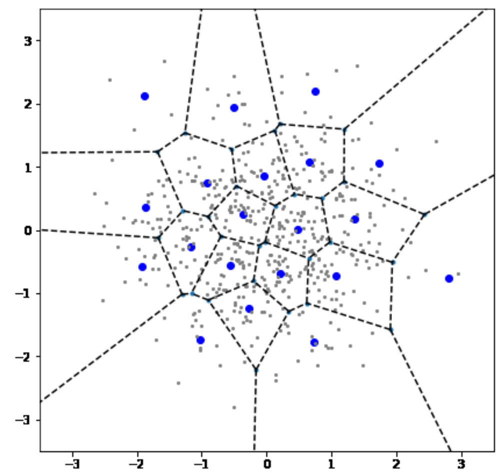

Figure 2: An optimal cluster center (blue points) for an observation $\{\eta_1, ..., \eta_{500}\} \overset{i.i.d}{\sim} \mathcal{N}(0, \mathbf{I}_2)$ (grey points).

For $p \in [1, +\infty)$, let $\mathcal{P}_p(\mathbb{R}^d)$ denote the set of all probability measures on $\mathbb{R}^d$ with a finite p^{th}-moment. Let $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$ and let $\Pi(\mu, \nu)$ denote the set of all probability measures on $(\mathbb{R}^d \times \mathbb{R}^d, Bor(\mathbb{R}^d)^{\otimes 2})$ with marginals μ and ν, where $Bor(\mathbb{R}^d)$ denotes the Borel σ-algebra on $\mathbb{R}^d$. For $p \geq 1$, the *L^p-Wasserstein distance* $\mathcal{W}_p$ on $\mathcal{P}_p(\mathbb{R}^d)$ is defined by

$$\mathcal{W}_p(\mu, \nu) = \Big(\inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \, \pi(dx, dy) \Big)^{\frac{1}{p}}$$
$$= \inf \Big\{ \Big[\mathbb{E} \, |X - Y|^p \Big]^{\frac{1}{p}}, \; X, Y : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathbb{R}^d, Bor(\mathbb{R}^d)) \text{ with } \mathbb{P}_X = \mu, \mathbb{P}_Y = \nu \Big\}.$$

The space $\mathcal{P}_p(\mathbb{R}^d)$ equipped with the Wasserstein distance $\mathcal{W}_p$ is a Polish space, i.e. is separable and complete (see Bolley, 2008). If $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$, then for any $q \leq p$, $\mathcal{W}_q(\mu, \nu) \leq \mathcal{W}_p(\mu, \nu)$.

With a slight abuse of notation, we define the distortion function for the optimal quantization as follows.

Definition 1 (Distortion function) *Let $K \in \mathbb{N}^*$ be the quantization level. Let $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. The (quadratic) distortion function $\mathcal{D}_{K,\mu}$ of μ at level K is defined by*

$$x = (x_1, ..., x_K) \in (\mathbb{R}^d)^K \longmapsto \mathcal{D}_{K,\mu}(x) = \int_{\mathbb{R}^d} \min_{1 \leq i \leq K} |\xi - x_i|^2 \, \mu(d\xi) = e^2_{K,\mu}(x).$$

For a fixed (known) probability distribution μ, its optimal quantizers can be computed by several algorithms such as the CLVQ algorithm (see e.g. Pagès (2015, Section 3.2)) or the Lloyd I algorithm (see e.g. Lloyd (1982), Kieffer (1982) and Pagès and Yu (2016)). However, another situation exists: the probability distribution μ is unknown but there exists a known sequence $(\mu_n)_{n\geq 1}$ converging in the Wasserstein distance to μ. A typical example is the empirical measure of an i.i.d. μ-distributed sequence random vectors (see (5) below). The empirical measure of non i.i.d. random vectors appears for example when dealing with the particle method associated to the McKean-Vlasov equations (see Liu, 2019, Section 7.1 and Section 7.5) or the simulation of the invariant measure of the diffusion process (see Lamberton and Pagès (2002) and Lemaire (2005, Chapter 4)). This leads us to study the consistency and the convergence rate of the optimal quantization for a $\mathcal{W}_p$-converging probability distribution sequence $(\mu_n)_{n\geq 1}$.

There exist several studies in the literature. The consistency of the optimal quantizers was first proved in Pollard (1982b).

Theorem (Pollard's Theorem)[1] *Let $\mu_n \in \mathcal{P}_2(\mathbb{R}^d), n \in \mathbb{N}^* \cup \{\infty\}$ with $\mathcal{W}_2(\mu_n, \mu_\infty) \to 0$ as $n \to +\infty$. Assume* $\text{card}\big(\text{supp}(\mu_n)\big) \geq K$, *for $n \in \mathbb{N}^* \cup \{+\infty\}$. For $n \geq 1$, let $x^{(n)} = \big(x_1^{(n)}, ..., x_K^{(n)}\big)$ be a K-optimal quantizer for μ_n, then the quantizer sequence $(x^{(n)})_{n\geq 1}$ is bounded in $\mathbb{R}^d$ and any limiting point of $(x^{(n)})_{n\geq 1}$, denoted by $x^{(\infty)}$, is an optimal quantizer of μ_∞.*

Let $\mu_n \in \mathcal{P}_2(\mathbb{R}^d)$, $n \in \mathbb{N} \cup \{\infty\}$ with $\mathcal{W}_2(\mu_n, \mu_\infty) \to 0$ as $n \to +\infty$. Let $x^{(n)}$ denote an optimal quantiser of μ_n. There are two ways to study the convergence rate of the optimal quantizers. The first way is to directly evaluate the distance between $x^{(n)}$ and $\text{argmin}\, \mathcal{D}_{K,\mu_\infty}$. The second way is called *the quantization performance*, defined by

$$\mathcal{D}_{K,\mu_\infty}(x^{(n)}) - \inf_{x\in(\mathbb{R}^d)^K} \mathcal{D}_{K,\mu_\infty}(x).$$

This quantity describes the distance between the optimal error of μ_∞ and the quantization error of $x^{(n)}$ considered as a quantizer of μ_∞ (even $x^{(n)}$ is obviously not "optimal" for μ_∞). Several results of convergence rate exist in the framework of the empirical measure. Let $X_1, ..., X_n, ...$ be μ-distributed i.i.d. random vectors defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let

$$\mu_n^\omega := \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i(\omega)} \tag{5}$$

be the empirical measure of μ. The almost sure convergence of $\mathcal{W}_2(\mu_n^\omega, \mu)$ has been proved in Pollard (1982b, Theorem 7). Let $x^{(n),\omega}$ denotes an optimal quantizer of μ_n^ω at level K. In Pollard (1982a), the author has proved that if μ has a unique optimal quantizer x at

1. In Pollard (1982b, Theorem 9), the author used

$$\mu_K \in \mathcal{P}(K) := \Big\{\nu \in \mathcal{P}_2(\mathbb{R}^d) \text{ such that } \text{card}\big(\text{supp}(\nu)\big) \leq K\Big\}$$

to represent a "quantizer" at level K. Such a quantizer μ_K is called "quadratic optimal" for a probability measure μ if $\mathcal{W}_2(\mu_K, \mu) = e^*_{K,\mu}$. We propose an alternative proof in Appendix A by using the usual representation of the quantizer $x \in (\mathbb{R}^d)^K$ but still call this theorem "Pollard's Theorem".

level K, then the convergence rate (convergence in distribution) of $\left|x^{(n),\omega} - x\right|$ is $\mathcal{O}(n^{-1/2})$ under appropriate conditions. Moreover, if μ has a support contained in $B(0,R)$, where $B(0,R)$ denotes the ball in $\mathbb{R}^d$ centered at 0 with radius R, an upper bound of the mean performance has been proved in Biau et al. (2008), shown as follows,

$$\mathbb{E}\,\mathcal{D}_{K,\,\mu}(x^{(n),\omega}) - \inf_{x\in(\mathbb{R}^d)^K} \mathcal{D}_{K,\,\mu}(x) \leq \frac{12K\cdot R^2}{\sqrt{n}}.$$

Note that there always exists an $\mathcal{A}$-measurable selection $\omega \mapsto x^{(n),\omega}$ relying on the Kuratowski and Ryll-Nardzewski measurable selection theorem (see e.g. Kuratowski and Ryll-Nardzewski (1965), Srivastava (1998, Section 5.2) and Graf (1982, Theorem 2.1)). We will always assume in what follows that we consider such a measurable selection. Otherwise all the stated results remain true by simply replacing the regular expectation by the inner expectation in the sense of Van Der Vaart and Wellner (1996).

In this paper, we extend the convergence results in Pollard (1982a) and in Biau et al. (2008) in two perspectives: first, we give an upper bound of the quantization performance

$$\mathcal{D}_{K,\mu_\infty}(x^{(n)}) - \inf_{x\in(\mathbb{R}^d)^K} \mathcal{D}_{K,\mu_\infty}(x)$$

and that of related optimal quantizers for any probability distribution sequence $(\mu_n)_{n\geq 1}$ converging in the Wasserstein distance. Then, we generalize the clustering performance results in Biau et al. (2008) to empirical measures in $\mathcal{P}_2(\mathbb{R}^d)$ possibly having an unbounded support.

Our main results are as follows. We obtain in Section 2 a non-asymptotic upper bound for the quantization performance: for every $n \in \mathbb{N}^*$,

$$\mathcal{D}_{K,\,\mu_\infty}(x^{(n)}) - \inf_{x\in(\mathbb{R}^d)^K} \mathcal{D}_{K,\,\mu_\infty}(x) \leq 4e^*_{K,\mu_\infty}\mathcal{W}_2(\mu_n,\mu_\infty) + 4\mathcal{W}_2^2(\mu_n,\mu_\infty). \tag{6}$$

Moreover, if $\mathcal{D}_{K,\mu_\infty}$ is twice differentiable at

$$F_K \coloneqq \left\{x = (x_1, ..., x_K) \in (\mathbb{R}^d)^K \;\middle|\; x_i \neq x_j,\ \text{if } i \neq j\right\} \tag{7}$$

and if the Hessian matrix $H_{\mathcal{D}_{K,\mu_\infty}}$ of $\mathcal{D}_{K,\mu_\infty}$ is positive definite in the neighbourhood of every K-level optimal quantizer $x^{(\infty)}$ of μ_∞ having the eigenvalues lower bounded by a $\lambda^* > 0$, then, for n large enough,

$$d\big(x^{(n)}, G_K(\mu_\infty)\big)^2 \leq \frac{8}{\lambda^*}e^*_{K,\mu_\infty}\cdot\mathcal{W}_2(\mu_n,\mu_\infty) + \frac{8}{\lambda^*}\cdot\mathcal{W}_2^2(\mu_n,\mu_\infty),$$

where $d(\xi, A) \coloneqq \min_{a\in A}|\xi - a|$ denotes the distance between a point $\xi \in \mathbb{R}^d$ and a set $A \subset \mathbb{R}^d$.

Several criterions for the positive definiteness of the Hessian matrix $H_{\mathcal{D}_{K,\mu}}$ of the distortion function $\mathcal{D}_{K,\mu}$ are established in Section 3. We show in Section 3.1 the conditions under which the distortion function $\mathcal{D}_{K,\mu}$ is twice differentiable in every $x \in F_K$ and give the exact formula of the Hessian matrix $H_{\mathcal{D}_{K,\mu}}$. Moreover, we also discuss several sufficient and necessary conditions for the positive definiteness of the Hessian matrix in dimension $d \geq 2$ and in dimension 1.

In Section 4, we give two upper bounds for the *clustering performance* $\mathbb{E}\,\mathcal{D}_{K,\,\mu}(x^{(n),\,\omega}) - \inf_{x\in(\mathbb{R}^d)^K}\mathcal{D}_{K,\,\mu}(x)$, where $x^{(n),\,\omega}$ is an optimal quantizer of μ_n^{ω} defined in (5). If $\mu\in\mathcal{P}_q(\mathbb{R}^d)$ for some $q>2$, a first upper bound is established in Proposition 13

$$\mathbb{E}\,\mathcal{D}_{K,\,\mu}(x^{(n),\omega}) - \inf_{x\in(\mathbb{R}^d)^K}\mathcal{D}_{K,\,\mu}(x)$$
$$\leq C_{d,q,\mu,K}\times\begin{cases} n^{-1/4}+n^{-(q-2)/2q} & \text{if } d<4 \text{ and } q\neq 4\\ n^{-1/4}\big(\log(1+n)\big)^{1/2}+n^{-(q-2)/2q} & \text{if } d=4 \text{ and } q\neq 4\\ n^{-1/d}+n^{-(q-2)/2q} & \text{if } d>4 \text{ and } q\neq d/(d-2)\end{cases},$$

where $C_{d,q,\mu,K}$ is a constant depending on d,q,μ and the quantization level K. This result is a direct application of the non-asymptotic upper bound (6) combined with results in Fournier and Guillin (2015) about the mean convergence rate of the empirical measure for the Wasserstein distance. If $d\geq 4$ and $q>\frac{2d}{d-2}$, this constant $C_{d,q,\mu,K}$ is roughly decreasing as $K^{-1/d}$ (see Remark 14). This upper bound is sharper in K compared with the upper bound (8) below, although it suffers from the curse of dimensionality.

Meanwhile, we establish another upper bound for the clustering performance in Theorem 15, which is sharper in n but increasing faster than linearly in K. This upper bound is

$$\mathbb{E}\,\mathcal{D}_{K,\,\mu}(x^{(n),\omega}) - \inf_{x\in(\mathbb{R}^d)^K}\mathcal{D}_{K,\,\mu}(x)\leq \frac{2K}{\sqrt{n}}\Big[r_{2n}^2+\rho_K(\mu)^2+2r_1\big(r_{2n}+\rho_K(\mu)\big)\Big], \tag{8}$$

where $r_n := \big\|\max_{1\leq i\leq n}|X_i|\,\big\|_2$ and $\rho_K(\mu)$ is the maximum radius of optimal quantizers for μ, defined by

$$\rho_K(\mu) := \max\Big\{\max_{1\leq k\leq K}|x_k^*|\,,\;\; (x_1^*,...,x_K^*) \text{ is an optimal quantizer of } \mu \text{ at level } K\Big\}. \tag{9}$$

In particular, we give a precise upper bound for $\mu=\mathcal{N}(m,\Sigma)$, the multidimensionnal normal distribution

$$\mathbb{E}\,\mathcal{D}_{K,\,\mu}(x^{(n),\omega}) - \inf_{x\in(\mathbb{R}^d)^K}\mathcal{D}_{K,\,\mu}(x)\leq C_\mu\cdot\frac{2K}{\sqrt{n}}\Big[1+\log n+\gamma_K\log K\big(1+\frac{2}{d}\big)\Big],$$

where $\limsup_K\gamma_K=1$ and $C_\mu = 12\cdot\Big[1\vee\log\big(2\int_{\mathbb{R}^d}\exp(\frac{1}{4}|\xi|^4)\mu(d\xi)\big)\Big]$. If $\mu=\mathcal{N}(0,\mathbf{I}_d)$, $C_\mu = 12(1+\frac{d}{2})\cdot\log 2$.

We start our discussion with a brief review on the properties of optimal quantization.

1.1. Classical Properties of Optimal Quantization

Let $G_K(\mu)=\operatorname{argmin}\mathcal{D}_{K,\mu}$ denote the set of all optimal quantizers at level K of μ and let $e_{K,\mu}^*$ denote the optimal quantization error of μ defined in (4).

Proposition 2 *Let $K\in\mathbb{N}^*$. Let $\mu\in\mathcal{P}_2(\mathbb{R}^d)$ and* $\operatorname{card}\big(\operatorname{supp}(\mu)\big)\geq K$.

(i) If $K\geq 2$, then $e_{K,\,\mu}^* < e_{K-1,\,\mu}^*$.

(ii) *(Existence and boundedness of optimal quantizers) The set $G_K(\mu)$ is nonempty and compact so that $\rho_K(\mu)$ defined in (9) is finite for any fixed K. Moreover, if $x = (x_1, ..., x_K)$ is an optimal quantizer of μ, then $x \in F_K$, where F_K is defined in (7).*

(iii) *If the support of μ, denoted by $\mathrm{supp}(\mu)$, is a compact, then for every optimal quantizer $x = (x_1, ..., x_K) \in G_K(\mu)$, its elements $x_k, 1 \leq k \leq K$ are contained in the closure of convex hull of $\mathrm{supp}(\mu)$, denoted by $\mathcal{H}_\mu := \overline{\mathrm{conv}\big(\mathrm{supp}(\mu)\big)}$.*

For the proof of Proposition 2-(i) and (ii), we refer to Graf and Luschgy (2000, Theorem 4.12) and for the proof of (iii) to Appendix B. Now we present an upper bound of the optimal quantization error (see Luschgy et al. (2008) and Pagès (2018, Theorem 5.2))).

Theorem (Non-asymptotic Zador's Theorem) *Let $\eta > 0$. If $\mu \in \mathcal{P}_{2+\eta}(\mathbb{R}^d)$, then for every quantization level K, there exists a constant $C_{d,\eta} \in (0, +\infty)$ which depends only on d and η such that*

$$e^*_{K,\mu} \leq C_{d,\eta} \cdot \sigma_{2+\eta}(\mu) K^{-1/d}, \tag{10}$$

where for $r \in (0, +\infty)$, $\sigma_r(\mu) = \min_{a \in \mathbb{R}^d} \big[\int_{\mathbb{R}^d} |\xi - a|^r \mu(d\xi) \big]^{1/r}$.

When μ has an unbounded support, we know from Pagès and Sagna (2012) that $\lim_K \rho_K(\mu) = +\infty$. The same paper also gives an asymptotic upper bound of ρ_K when μ has a polynomial tail or a hyper-exponential tail.

Theorem (Pagès and Sagna, 2012, Theorem 1.2) *Let $\mu \in \mathcal{P}_p(\mathbb{R}^d)$ be absolutely continuous with respect to the Lebesgue measure λ_d on $\mathbb{R}^d$ and let f denote its density function.*

(i) Polynomial tail. *For $p \geq 2$, if μ has a c-th polynomial tail with $c > d + p$ in the sense that there exists $\tau > 0, \beta \in \mathbb{R}$ and $A > 0$ such that $\forall \xi \in \mathbb{R}^d, |\xi| \geq A \Longrightarrow f(\xi) = \frac{\tau}{|\xi|^c} (\log |\xi|)^\beta$, then*

$$\lim_K \frac{\log \rho_K}{\log K} = \frac{p+d}{d(c-p-d)}. \tag{11}$$

(ii) Hyper-exponential tail. *If μ has a (ϑ, κ)-hyper-exponential tail in the sense that there exists $\tau > 0, \kappa, \vartheta > 0, c > -d$ and $A > 0$ such that $\forall \xi \in \mathbb{R}^d, |\xi| \geq A \Longrightarrow f(\xi) = \tau |\xi|^c e^{-\vartheta |\xi|^\kappa}$, then*

$$\limsup_K \frac{\rho_K}{(\log K)^{1/\kappa}} \leq 2\vartheta^{-1/\kappa}\Big(1 + \frac{2}{d}\Big)^{1/\kappa}. \tag{12}$$

Furthermore, if $d = 1$, $\lim_K \frac{\rho_K}{(\log K)^{1/\kappa}} = \big(\frac{3}{\vartheta}\big)^{1/\kappa}$.

We give now the definition of the *radially controlled* distribution, which will be useful to *control* the convergence rate of the density function $f(x)$ to 0 when x converges in every direction to infinity.

Definition 3 *Let $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ be absolutely continuous with respect to the Lebesgue measure λ_d on $\mathbb{R}^d$ having a continuous density function f. We call μ is k-radially controlled on $\mathbb{R}^d$ if there exists $A > 0$ and a continuous non-increasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\forall \xi \in \mathbb{R}^d, |\xi| \geq A, \qquad f(\xi) \leq g(|\xi|) \text{ and } \int_{\mathbb{R}_+} x^{d-1+k} g(x) dx < +\infty.$$

Note that the c-th polynomial tail with $c > k + d$ and the hyper-exponential tail are sufficient conditions to satisfy the k-radially controlled assumption. A typical example of hyper-exponential tail is the multidimensional normal distribution $\mathcal{N}(m, \Sigma)$.

For $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and for every $K \in \mathbb{N}^*$, we have

$$\|e_{K,\mu} - e_{K,\nu}\|_{\sup} \coloneqq \sup_{x\in(\mathbb{R}^d)^K} |e_{K,\mu}(x) - e_{K,\nu}(x)| \le \mathcal{W}_2(\mu, \nu)$$

by a simple application of the triangle inequality for the L^2-norm (see e.g. Graf and Luschgy, 2000, Formula (4.4) and Lemma 3.4). Hence, if $(\mu_n)_{n\ge1}$ is a sequence in $\mathcal{P}_2(\mathbb{R}^d)$ converging for the $\mathcal{W}_2$-distance to $\mu_\infty \in \mathcal{P}_2(\mathbb{R}^d)$, then for every $K \in \mathbb{N}^*$,

$$\|e_{K,\mu_n} - e_{K,\mu_\infty}\|_{\sup} \le \mathcal{W}_2(\mu_n, \mu_\infty) \xrightarrow{n\to+\infty} 0. \tag{13}$$

2. General Case

In this section, we first establish in Theorem 4 a non-asymptotic upper bound of the quantization performance $\mathcal{D}_{K,\mu_\infty}(x^{(n)}) - \inf_{x\in(\mathbb{R}^d)^K} \mathcal{D}_{K,\mu_\infty}(x)$. Then we discuss the convergence rate of the optimal quantizer sequence in Theorem 5.

Theorem 4 (Non-asymptotic upper bound for the quantization performance) *Let $K \in \mathbb{N}^*$ be the quantization level. For every $n \in \mathbb{N}^* \cup \{\infty\}$, let $\mu_n \in \mathcal{P}_2(\mathbb{R}^d)$ with $\mathrm{card}\big(\mathrm{supp}(\mu_n)\big) \ge K$. Assume that $\mathcal{W}_2(\mu_n, \mu_\infty) \to 0$ as $n \to +\infty$. For every $n \in \mathbb{N}^*$, let $x^{(n)}$ be an optimal quantizer of μ_n. Then*

$$\mathcal{D}_{K,\mu_\infty}(x^{(n)}) - \inf_{x\in(\mathbb{R}^d)^K} \mathcal{D}_{K,\mu_\infty}(x) \le 4e^*_{K,\mu_\infty}\mathcal{W}_2(\mu_n, \mu_\infty) + 4\mathcal{W}_2^2(\mu_n, \mu_\infty),$$

where e^*_{K,μ_∞} is the optimal error of μ_∞ at level K defined in (4).

Proof Let $x^{(\infty)}$ be an optimal quantizer of μ_∞. Remark that here we do not need that $x^{(\infty)}$ is the limit of $x^{(n)}$. First, we have (see e.g. Györfi, 2002, Corollary 4.1)

$$\begin{aligned} e_{K,\mu_\infty}(x^{(n)}) - e^*_{K,\mu_\infty} &= e_{K,\mu_\infty}(x^{(n)}) - e_{K,\mu_n}(x^{(n)}) + e_{K,\mu_n}(x^{(n)}) - e_{K,\mu_\infty}(x^{(\infty)}) \\ &\le 2\,\|e_{K,\mu_\infty} - e_{K,\mu_n}\|_{\sup} \le 2\mathcal{W}_2(\mu_n, \mu_\infty), \end{aligned} \tag{14}$$

where the first inequality is due to the fact that for any $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ with respective K-level optimal quantizers x^μ and x^ν, if $e_{K,\mu}(x^\mu) \ge e_{K,\nu}(x^\nu)$, we have

$$|e_{K,\mu}(x^\mu) - e_{K,\nu}(x^\nu)| = e_{K,\mu}(x^\mu) - e_{K,\nu}(x^\nu) \le e_{K,\mu}(x^\nu) - e_{K,\nu}(x^\nu) \le \|e_{K,\mu_\infty} - e_{K,\mu_n}\|_{\sup}.$$

If $e_{K,\mu}(x^\mu) \le e_{K,\nu}(x^\nu)$, we have the same inequality by the same reasoning.

Moreover,

$$\begin{aligned} \mathcal{D}_{K,\mu_\infty}(x^{(n)}) - \inf_{x\in(\mathbb{R}^d)^K} \mathcal{D}_{K,\mu_\infty}(x) &= \mathcal{D}_{K,\mu_\infty}(x^{(n)}) - \mathcal{D}_{K,\mu_\infty}(x^{(\infty)}) \\ &\le \big[e_{K,\mu_\infty}(x^{(n)}) + e_{K,\mu_\infty}(x^{(\infty)})\big]\big(e_{K,\mu_\infty}(x^{(n)}) - e_{K,\mu_\infty}(x^{(\infty)})\big) \end{aligned}$$

$$
\begin{aligned}
&\le 2\big[e_{K,\mu_\infty}(x^{(n)}) - e_{K,\mu_\infty}(x^{(\infty)}) + 2e_{K,\mu_\infty}(x^{(\infty)})\big]\cdot \mathcal{W}_2(\mu_n,\mu_\infty) \quad \big(\text{by } (14)\big)\\
&\le 4\big[\mathcal{W}_2(\mu_n,\mu_\infty) + e^*_{K,\mu_\infty}\big]\cdot \mathcal{W}_2(\mu_n,\mu_\infty) \quad \big(\text{by } (14)\big)\\
&\le 4e^*_{K,\mu_\infty}\mathcal{W}_2(\mu_n,\mu_\infty) + 4\mathcal{W}_2^2(\mu_n,\mu_\infty). \qquad \blacksquare
\end{aligned}
$$

Let $B(x,r)$ denote the ball centered at x with radius r. Recall that $F_K \coloneqq \big\{x = (x_1,...,x_K) \in (\mathbb{R}^d)^K \;\big|\; x_i \neq x_j, \text{ if } i \neq j\big\}$. Remark that if $x \in F_K$, then every $y \in B\big(x, \frac{1}{3}\min_{1\le i,j\le K, i\neq j}|x_i - x_j|\big)$ still lies in F_K. In the following theorem, we give an estimate of the convergence rate of the optimal quantizer sequence $x^{(n)}, n \in \mathbb{N}^*$.

Theorem 5 (Convergence rate of optimal quantizers) *Let $K \in \mathbb{N}^*$ be the quantization level. For every $n \in \mathbb{N}^* \cup \{\infty\}$, let $\mu_n \in \mathcal{P}_2(\mathbb{R}^d)$ with $\mathrm{card}\big(\mathrm{supp}(\mu_n)\big) \ge K$. Assume that $\mathcal{W}_2(\mu_n,\mu_\infty) \to 0$ as $n \to +\infty$. For every $n \in \mathbb{N}^*$, let $x^{(n)}$ be an optimal quantizer of μ_n and let $G_K(\mu_\infty)$ denote the set of all optimal quantizers of μ_∞. If the following assumptions hold*

(a) the distortion function $\mathcal{D}_{K,\mu_\infty}$ is twice differentiable at every $x \in F_K$;

(b) $\mathrm{card}\big(G_K(\mu_\infty)\big) < +\infty$;

(c) for every $x^{(\infty)} \in G_K(\mu_\infty)$, the Hessian matrix of $\mathcal{D}_{K,\mu_\infty}$, denoted by $H_{\mathcal{D}_{K,\mu_\infty}}$, is positive definite in the neighbourhood of $x^{(\infty)}$ having eigenvalues lower bounded by some $\lambda^* > 0$,

then, for n large enough,

$$
d\big(x^{(n)}, G_K(\mu_\infty)\big)^2 \le \frac{8}{\lambda^*} e^*_{K,\mu_\infty}\cdot \mathcal{W}_2(\mu_n,\mu_\infty) + \frac{8}{\lambda^*}\cdot \mathcal{W}_2^2(\mu_n,\mu_\infty).
$$

Remark 6 *Section 3 provides a detailed discussion of the conditions in Theorem 5 and their relation between each other.*
(1) First, in Section 3, we establish that if μ_∞ is 1-radially controlled, then its distortion function $\mathcal{D}_{K,\mu_\infty}$ is twice continuously differentiable at every $x \in F_K$ and give an exact formula of the Hessian matrix $H_{\mathcal{D}_{K,\mu_\infty}}(x)$ in Proposition 8. Thus, one may obtain Condition (c) either by an explicit computation or by numerical methods. Moreover, if $H_{\mathcal{D}_{K,\mu}}$ is positive definite at $x \in F_K$, it is also positive definite in its neighbourhood. In Section 3.2, we establish several sufficient conditions for the positive definiteness of the Hessian matrix $H_{\mathcal{D}_{K,\mu_\infty}}$ in the neighbourhood of $x^{(\infty)} \in G_K(\mu_\infty)$ in one dimension.
(2) If the distribution μ_∞ is 1-radially controlled, a necessary condition for Condition (c) is Condition (b) (see Lemma 9). Thus, if $\mathrm{card}\big(G_K(\mu_\infty)\big) = +\infty$, it is more reasonable to consider the non-asymptotic upper bound of the performance (Theorem 4) to study the convergence rate of the optimal quantization. A typical example is the standard multidimensional normal distribution $\mu_\infty = \mathcal{N}(0, I_d)$: it is 1-radially controlled and any rotation of an optimal quantizer x is still optimal so that $\mathrm{card}\big(G_K(\mu_\infty)\big) = +\infty$.

Proof [Proof of Theorem 5] Since the quantization level K is fixed throughout the proof, we will drop the subscripts K and μ of the distortion function $\mathcal{D}_{K,\mu}$ and we will denote by $\mathcal{D}_n$ (*respectively,* $\mathcal{D}_\infty$) the distortion function of μ_n (*resp.* μ_∞).

After Pollard's theorem, $(x^{(n)})_{n\in\mathbb{N}^*}$ is bounded and any limiting point of $x^{(n)}$ lies in $G_K(\mu_\infty)$. We may assume that, up to the extraction of a subsequence of $x^{(n)}$, still denoted by $x^{(n)}$, we have $x^{(n)} \to x^{(\infty)} \in G_K(\mu_\infty)$. Hence $d\big(x^{(n)}, G_K(\mu_\infty)\big) \le \big|x^{(n)} - x^{(\infty)}\big|$.

Proposition 2 implies that $x^{(\infty)} \in F_K$. As $\mathcal{D}_\infty$ is twice differentiable at $x^{(\infty)}$, the second order Taylor expansion of $\mathcal{D}_\infty$ at $x^{(\infty)}$ reads:

$$\mathcal{D}_\infty(x^{(n)}) = \mathcal{D}_\infty(x^{(\infty)}) + \big\langle \nabla\mathcal{D}_\infty(x^{(\infty)}) \mid x^{(n)} - x^{(\infty)}\big\rangle + \frac{1}{2}H_{\mathcal{D}_\infty}(\zeta^{(n)})(x^{(n)} - x^{(\infty)})^{\otimes 2},$$

where $H_{\mathcal{D}_\infty}$ denotes the Hessian matrix of $\mathcal{D}_\infty$, $\zeta^{(n)}$ lies in the geometric segment $(x^{(n)}, x^{(\infty)})$ and for a matrix A and a vector u, $Au^{\otimes 2}$ stands for $u^T Au$.

As $x^{(\infty)} \in G_K(\mu_\infty) = \operatorname{argmin}\mathcal{D}_\infty$ and $\operatorname{card}\big(\operatorname{supp}(\mu_\infty)\big) \ge K$, one has $\nabla\mathcal{D}_\infty(x^{(\infty)}) = 0$. Hence

$$\mathcal{D}_\infty(x^{(n)}) - \mathcal{D}_\infty(x^{(\infty)}) = \frac{1}{2}H_{\mathcal{D}_\infty}(\zeta^{(n)})(x^{(n)} - x^{(\infty)})^{\otimes 2}.$$

It follows from Theorem 4 that

$$\begin{aligned} H_{\mathcal{D}_\infty}(\zeta^{(n)})(x^{(n)} - x^{(\infty)})^{\otimes 2} &= 2\big(\mathcal{D}_\infty(x^{(n)}) - \mathcal{D}_\infty(x^{(\infty)})\big) \\ &\le 8e^*_{K,\mu_\infty}\mathcal{W}_2(\mu_n, \mu_\infty) + 8\mathcal{W}_2^2(\mu_n, \mu_\infty). \end{aligned}$$

By Condition (c), $H_{\mathcal{D}_\infty}$ is assumed to be positive definite in the neighbourhood of all $x^{(\infty)} \in G_K(\mu_\infty)$ having eigenvalues lower bounded by some $\lambda^* > 0$. As $\zeta^{(n)}$ lies in the geometric segment $(x^{(n)}, x^{(\infty)})$ and $x^{(n)} \to x^{(\infty)}$, there exists an $n_0(x^{(\infty)})$ such that for all $n \ge n_0$, $H_{\mathcal{D}_\infty}(\zeta^{(n)})$ is a positive definite matrix. It follows that, for $n \ge n_0$,

$$\begin{aligned} \lambda^* \left|x^{(n)} - x^{(\infty)}\right|^2 &\le H_{\mathcal{D}_\infty}(\zeta^{(n)})(x^{(n)} - x^{(\infty)})^{\otimes 2} \\ &\le 8e^*_{K,\mu_\infty}\mathcal{W}_2(\mu_n, \mu_\infty) + 8\mathcal{W}_2^2(\mu_n, \mu_\infty). \end{aligned}$$

Thus, one can directly conclude by multiplying at each side of the above inequality by $\frac{1}{\lambda^*}$. ■

Based on conditions in Theorem 5, if we know the exact limit of the optimal quantizer sequence $x^{(n)}$, we have the following result whose proof is similar to that of Theorem 5.

Corollary 7 *Let $K \in \mathbb{N}^*$ be the quantization level. For every $n \in \mathbb{N}^* \cup \{\infty\}$, let $\mu_n \in \mathcal{P}_2(\mathbb{R}^d)$ with $\operatorname{card}\big(\operatorname{supp}(\mu_n)\big) \ge K$. Assume that $\mathcal{W}_2(\mu_n, \mu_\infty) \to 0$ as $n \to +\infty$. Let $x^{(n)} \in \operatorname{argmin}\ \mathcal{D}_{K,\mu_n}$ such that $\lim_n x^{(n)} \to x^{(\infty)}$. If the Hessian matrix of $\mathcal{D}_{K,\mu_\infty}$ is positive definite in the neighbourhood of $x^{(\infty)}$, then, for n large enough,*

$$\left|x^{(n)} - x^{(\infty)}\right|^2 \le C^{(1)}_{\mu_\infty} \cdot \mathcal{W}_2(\mu_n, \mu_\infty) + C^{(2)}_{\mu_\infty} \cdot \mathcal{W}_2^2(\mu_n, \mu_\infty),$$

where $C^{(1)}_{\mu_\infty}$ and $C^{(2)}_{\mu_\infty}$ are real constants only depending on μ_∞.

3. Hessian Matrix $H_{\mathcal{D}_{K,\mu}}$ of the Distortion Function $\mathcal{D}_{K,\mu}$

Let $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ with $\operatorname{card}\big(\operatorname{supp}(\mu)\big) \ge K$ and let x^* be an optimal quantizer of μ at level K. In Section 3.1, we show conditions under which the distortion function $\mathcal{D}_{K,\mu}$ is twice differentiable and give the exact formula of its Hessian matrix $H_{\mathcal{D}_{K,\mu}}$. In Section 3.2, we give several criterions for the positive definiteness of the Hessian matrix $H_{\mathcal{D}_{K,\mu}}$ in the neighbourhood of an optimal quantizer x^* in dimension 1.

3.1. Hessian Matrix $H_{\mathcal{D}_{K,\mu}}$ on $\mathbb{R}^d$

If μ is absolutely continuous with respect to the Lebesgue measure λ_d on $\mathbb{R}^d$ with the density function f, then the distortion function $\mathcal{D}_{K,\mu}$ is differentiable (see Pagès, 1998) at all point $x=(x_1,...,x_K)\in F_K$ with

$$\frac{\partial \mathcal{D}_{K,\mu}}{\partial x_i}(x)=2\int_{V_i(x)}(x_i-\xi)f(\xi)\lambda_d(d\xi),\quad \text{for } i=1,...,K. \tag{15}$$

In the following Proposition, we give a criterion for the twice differentiability of the distortion function $\mathcal{D}_{K,\mu}$.

Proposition 8 *Let $\mu\in\mathcal{P}_2(\mathbb{R}^d)$ be absolutely continuous with respect to the Lebesgue measure λ_d on $\mathbb{R}^d$ with a continuous density function f. If μ is 1-radially controlled, then*

(i) the distortion function $\mathcal{D}_{K,\mu}$ is twice differentiable at every $x\in F_K$ and the Hessian matrix $H_{\mathcal{D}_{K,\mu}}(x)=\Big[\frac{\partial^2\mathcal{D}_{K,\mu}}{\partial x_j\partial x_i}(x)\Big]_{1\le i\le j\le K}$ is defined by

$$\frac{\partial^2 \mathcal{D}_{K,\mu}}{\partial x_j\partial x_i}(x)=2\int_{V_i(x)\cap V_j(x)}(x_i-\xi)\otimes(x_j-\xi)\cdot\frac{1}{|x_j-x_i|}f(\xi)\lambda_x^{ij}(d\xi),\quad \text{if } j\neq i, \tag{16}$$

$$\frac{\partial^2 \mathcal{D}_{K,\mu}}{\partial x_i^2}(x)=2\Big[\mu\big(V_i(x)\big)\mathrm{I}_d-\sum_{\substack{i\neq j\\1\le j\le K}}\int_{V_i(x)\cap V_j(x)}(x_i-\xi)\otimes(x_i-\xi)\cdot\frac{1}{|x_j-x_i|}f(\xi)\lambda_x^{ij}(d\xi)\Big], \tag{17}$$

where in (16) and (17), $u\otimes v:=[u^iv^j]_{1\le i,j\le d}$ for any two vectors $u=(u^1,...,u^d)$ and $v=(v^1,...,v^d)$ in $\mathbb{R}^d$;

(ii) every element $\frac{\partial^2\mathcal{D}_{K,\mu}}{\partial x_j\partial x_i}$ of the Hessian matrix $H_{\mathcal{D}_{K,\mu}}$ is continuous at every $x\in F_K$.

The proof of Proposition 8 is postponed to Appendix C. The following lemma shows that under the condition of Proposition 8, Condition (c) in Theorem 5 implies Condition (b).

Lemma 9 *Let $\mu\in\mathcal{P}_2(\mathbb{R}^d)$ be absolutely continuous with the respect to the Lebesgue measure λ_d on $\mathbb{R}^d$ with a continuous density function f. If μ_∞ is 1-radially controlled and $\mathrm{card}\big(G_K(\mu_\infty)\big)=+\infty$, then there exists a point $x\in G_K(\mu_\infty)$ such that the Hessian matrix $H_{\mathcal{D}_{K,\mu_\infty}}$ of $\mathcal{D}_{K,\mu_\infty}$ at x has an eigenvalue 0.*

Proof We denote by $H_{\mathcal{D}_\infty}$ instead of $H_{\mathcal{D}_{K,\mu_\infty}}$ to simplify the notation. Proposition 2 implies that $G_K(\mu_\infty)$ is a compact set. If $\mathrm{card}\big(G_K(\mu_\infty)\big)=+\infty$, there exists $x,x^{(k)}\in G_K(\mu_\infty), k\in\mathbb{N}^*$ such that $x^{(k)}\to x$ when $k\to+\infty$. Set $u_k:=\frac{x^{(k)}-x}{|x^{(k)}-x|}$, $k\ge1$, then we have $|u_k|=1$ for all $k\in\mathbb{N}^*$. Hence, there exists a subsequence $\varphi(k)$ of k such that $u_{\varphi(k)}$ converges to some $\widetilde{u}$ with $|\widetilde{u}|=1$.

The Taylor expansion of $\mathcal{D}_{K,\mu_\infty}$ at x reads:

$$\mathcal{D}_{K,\mu_\infty}(x^{\varphi(k)})=\mathcal{D}_{K,\mu_\infty}(x)+\big\langle\nabla\mathcal{D}_{K,\mu_\infty}(x)\;\big|\;x^{\varphi(k)}-x\big\rangle+\frac{1}{2}H_{\mathcal{D}_\infty}(\zeta^{\varphi(k)})(x^{\varphi(k)}-x)^{\otimes 2},$$

where $\zeta^{\varphi(k)}$ lies in the geometric segment $(x^{\varphi(k)}, x)$. Since $x, x^{(k)}, k \in \mathbb{N}^* \in G_K(\mu_\infty)$, then $\nabla\mathcal{D}_{K,\mu_\infty}(x) = 0$ and for any $k \in \mathbb{N}^*$, $\mathcal{D}_{K,\mu_\infty}(x^{\varphi(k)}) = \mathcal{D}_{K,\mu_\infty}(x)$. Hence, for any $k \in \mathbb{N}^*$, $H_{\mathcal{D}_\infty}(\zeta^{\varphi(k)})(x^{\varphi(k)} - x)^{\otimes 2} = 0$. Consequently, for any $k \in \mathbb{N}^*$,

$$H_{\mathcal{D}_\infty}(\zeta^{\varphi(k)})\Big(\frac{x^{\varphi(k)} - x}{\big|x^{\varphi(k)} - x\big|}\Big)^{\otimes 2} = 0.$$

Thus we have $H_{\mathcal{D}_\infty}(x)\widetilde{u}^{\otimes 2} = 0$ by letting $k \to +\infty$, so that $H_{\mathcal{D}_\infty}(x)$ has an eigenvalue 0. ■

3.2. A Criterion for Positive Definiteness of $H_{\mathcal{D}_\infty}(x^*)$ in 1-dimension

Let $\mu \in \mathcal{P}_2(\mathbb{R})$ with $\mathrm{card}\big(\mathrm{supp}(\mu)\big) \geq K$. Assume that μ is absolutely continuous with respect to the Lebesgue measure having a density function f. In the one-dimensional case, it is useful to point out a sufficient condition for the uniqueness of optimal quantizer. A probability distribution μ is called *strongly unimodal* if its density function f satisfies that $I = \{f > 0\}$ is an open (possibly unbounded) interval and $\log f$ is concave on I. Let $F_K^+ := \big\{x = (x_1, ..., x_K) \in \mathbb{R}^K \mid -\infty < x_1 < x_2 < ... < x_K < +\infty\big\}$.

Lemma 10 *For $K \in \mathbb{N}^*$, if μ is strongly unimodal with* $\mathrm{card}\big(\mathrm{supp}(\mu)\big) \geq K$*, then there is only one stationary (then optimal) quantizer of level K in F_K^+.*

We refer to Kieffer (1983), Trushkin (1982), Bouton and Pagès (1993) and Graf and Luschgy (2000, Theorem 5.1) for the proof of Lemma 10 and for more details.

Given a K-tuple $x = (x_1, ..., x_K) \in F_K^+$, the Voronoi region $V_i(x)$ can be explicitly written: $V_1(x) = (-\infty, \frac{x_1+x_2}{2}]$, $V_K(x) = [\frac{x_{K-1}+x_K}{2}, +\infty)$ and $V_i(x) = [\frac{x_{i-1}+x_i}{2}, \frac{x_i+x_{i+1}}{2}]$ for $i = 2, ..., K-1$. For all $x \in F_K^+$, $\mathcal{D}_{K,\mu}$ is differentiable at x and by (15) and

$$\nabla\mathcal{D}_{K,\mu}(x) = \left[\int_{V_i(x)} 2(x_i - \xi) f(\xi) d\xi\right]_{i=1,\dots,K}.$$

Therefore, as $\nabla\mathcal{D}_{K,\mu}(x^*) = 0$, one can solve the optimal quantizer $x^* \in F_K^+$ as follows,

$$x_i^* = \frac{\int_{V_i(x^*)} \xi f(\xi) d\xi}{\mu\big(V_i(x^*)\big)}, \quad \text{for } i = 1, ..., K. \tag{18}$$

For any $x \in F_K^+$, the Hessian matrix $H_{\mathcal{D}_{K,\mu}}$ of $\mathcal{D}_{K,\mu}$ at x is a tridiagonal symmetry matrix and can be calculated as follows,

$$H_{\mathcal{D}_{K,\mu}}(x) = \begin{pmatrix} A_1 - B_{1,2} & -B_{1,2} & & & & \\ & \ddots & & & & \\ & -B_{i-1,i} & A_i - B_{i-1,i} - B_{i,i+1} & -B_{i,i+1} & & \\ & & & \ddots & & \\ & & & & -B_{K-1,K} & A_K - B_{K-1,K} \end{pmatrix}, \tag{19}$$

where $A_i = 2\mu\big(C_i(x)\big)$ for $1 \leq i \leq K$ and $B_{i,j} = \frac{1}{2}(x_j - x_i)f(\frac{x_i+x_j}{2})$ for $1 \leq i < j \leq K$. Let F_μ be the cumulative distribution function of μ, then

$$
\begin{aligned}
&A_1 = 2\mu\big(C_1(x)\big) = 2F_\mu\Big(\frac{x_1+x_2}{2}\Big),\\
&A_i = 2\mu\big(C_i(x)\big) = 2\Big[F_\mu\Big(\frac{x_{i+1}+x_i}{2}\Big) - F_\mu\Big(\frac{x_{i-1}+x_i}{2}\Big)\Big], \quad \text{for } \; i = 2, ..., K-1,\\
&A_K = 2\mu\big(C_K(x)\big) = 2\Big[1 - F_\mu\Big(\frac{x_{K-1}+x_K}{2}\Big)\Big].
\end{aligned}
$$

Then the continuity of each term in the matrix $H_{\mathcal{D}_{K,\mu}}(x)$ can be directly derived from the continuity of F_μ.

For $1 \leq i \leq K$, we define $L_i(x) \coloneqq \sum_{j=1}^{K} \frac{\partial^2 \mathcal{D}_{K,\mu}}{\partial x_i \partial x_j}(x)$. The following proposition gives sufficient conditions to obtain the positive definiteness of $H_{\mathcal{D}_{K,\mu}}(x^*)$.

Proposition 11 *Let $\mu \in \mathcal{P}_2(\mathbb{R})$ with $\mathrm{card}\big(\mathrm{supp}(\mu)\big) \geq K$. Assume that μ is absolutely continuous with respect to the Lebesgue measure having a density function f. Any of the following two conditions implies the positive definiteness of $H_{\mathcal{D}_{K,\mu}}(x^*)$,*

(i) μ is the uniform distribution,

(ii) f is differentiable and $\log f$ is strictly concave.

In particular, (ii) also implies that $L_i(x^*) > 0$, $i = 1, ..., K$.

Proposition 11 is proved in Appendix D. Remark that, under the conditions of Proposition 11, μ is strongly unimodal so that there is exactly one optimal quantizer in F_K^+ for μ at level K. The conditions in Proposition 11 directly imply the following convergence rate results.

Theorem 12 *Let $K \in \mathbb{N}^*$ be the quantization level. For every $n \in \mathbb{N}^* \cup \{\infty\}$, let $\mu_n \in \mathcal{P}_2(\mathbb{R})$ with $\mathrm{card}\big(\mathrm{supp}(\mu_n)\big) \geq K$ be such that $\mathcal{W}_2(\mu_n, \mu_\infty) \to 0$ as $n \to +\infty$. Assume that μ_∞ is absolutely continuous with respect to the Lebesgue measure, written $\mu_\infty(d\xi) = f(\xi)d\xi$. Let $x^{(n)}$ be an optimal quantizer of μ_n converging to $x^{(\infty)}$. Then any one of the following two conditions*

(i) μ_∞ is the uniform distribution

(ii) f is differentiable and $\log f$ is strictly concave

implies the existence of constants $C_{\mu_\infty}^{(1)}$ and $C_{\mu_\infty}^{(2)}$ only depending on μ_∞ such that for n large enough,

$$
\Big|x^{(n)} - x^{(\infty)}\Big|^2 \leq C_{\mu_\infty}^{(1)} \cdot \mathcal{W}_2(\mu_n, \mu_\infty) + C_{\mu_\infty}^{(2)} \cdot \mathcal{W}_2^2(\mu_n, \mu_\infty).
$$

Proof Let $\mathcal{D}_{K,\mu_\infty}$ denote the distortion function of μ_∞ and let $H_{\mathcal{D}_\infty}$ denote the Hessian matrix of $\mathcal{D}_{K,\mu_\infty}$.

(i) Let $g_k(x)$ be the k-th leading principal minor of $H_{\mathcal{D}_\infty}(x)$ defined in (19), then $g_k(x), k = 1, ..., K$, are continuous functions in x since every element in this matrix is continuous. Proposition 11 implies $g_k(x^{(\infty)}) > 0$, thus there exists $r > 0$ such that for every $x \in B(x^{(\infty)}, r)$, $g_k(x^{(\infty)}) > 0$ so that $H_{\mathcal{D}_\infty}(x)$ is positive definite. What remains can be directly proved by Corollary 7.

(ii) The function $L_i(x) \coloneqq \sum_{j=1}^{K} \frac{\partial^2 \mathcal{D}_{K,\mu_\infty}}{\partial x_i \partial x_j}(x)$ is continuous on x and Proposition 11 implies that $L_i(x^{(\infty)}) > 0$. Hence, there exists $r > 0$ such that $\forall x \in B(x^{(\infty)}, r)$, $L_i(x) > 0$. From (19), one can remark that the i-th diagonal elements in $H_{\mathcal{D}_\infty}(x)$ is always larger than $L_i(x)$ for any $x \in \mathbb{R}^K$, then after Gershgorin Circle theorem, we derive that $H_{\mathcal{D}_\infty}(x)$ is positive definite for every $x \in B(x^{(\infty)}, r)$. What remains can be directly proved by Corollary 7. ■

4. Empirical Measure Case

Let $K \in \mathbb{N}^*$ be the quantization level. Let $\mu \in \mathcal{P}_{2+\varepsilon}(\mathbb{R}^d)$ for some $\varepsilon > 0$ and $\text{card}\big(\text{supp}(\mu)\big) \geq K$. Let X be a random variable with distribution μ and let $(X_n)_{n\geq 1}$ be a sequence of independent identically distributed $\mathbb{R}^d$-valued random variables with probability distribution μ. The empirical measure is defined for every $n \in \mathbb{N}^*$ by

$$\mu_n^\omega \coloneqq \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i(\omega)}, \quad \omega \in \Omega, \tag{20}$$

where δ_a is the Dirac mass at a. For $n \geq 1$, let $x^{(n),\omega}$ be an optimal quantizer of μ_n^ω. The superscript ω is to emphasize that both μ_n^ω and $x^{(n),\omega}$ are random and we will drop ω when there is no ambiguity. We cite two results of the convergence of $\mathcal{W}_2(\mu_n^\omega, \mu)$ among so many researches in this topic: the a.s. convergence in Pollard (1982b, Theorem 7) and the L^p-convergence rate of $\mathcal{W}_p(\mu_n^\omega, \mu)$ in Fournier and Guillin (2015).

Theorem (Fournier and Guillin, 2015, Theorem 1) *Let $p > 0$ and let $\mu \in \mathcal{P}_q(\mathbb{R}^d)$ for some $q > p$. Let μ_n^ω denote the empirical measure of μ defined in (20). There exists a constant C only depending on p, d, q such that, for all $n \geq 1$,*

$$\mathbb{E}\Big(\mathcal{W}_p^p(\mu_n^\omega, \mu)\Big) \leq C M_q^{p/q}(\mu) \times \begin{cases} n^{-1/2} + n^{-(q-p)/q} & \text{if } p > d/2 \text{ and } q \neq 2p \\ n^{-1/2}\log(1+n) + n^{-(q-p)/q} & \text{if } p = d/2 \text{ and } q \neq 2p \\ n^{-p/d} + n^{-(q-p)/q} & \text{if } p \in (0, d/2) \text{ and } q \neq d/(d-p) \end{cases}, \tag{21}$$

where $M_q(\mu) = \int_{\mathbb{R}^d} |\xi|^q \, \mu(d\xi)$.

Let $\mathcal{D}_{K,\mu}$ denote the distortion function of μ and let $\mathcal{D}_{K,\mu_n}$ denote the distortion fuction of μ_n^ω for any $n \in \mathbb{N}^*$. Recall by Definition 1 that for $c = (c_1, ..., c_K) \in (\mathbb{R}^d)^K$,

$$\mathcal{D}_{K,\mu}(c) = \mathbb{E} \min_{1\leq k\leq K} |X - c_k|^2 = \mathbb{E}\Big[|X|^2 + \min_{1\leq k\leq K}\big(-2\langle X|c_k\rangle + |c_k|^2\big)\Big],$$

$$\text{and } \mathcal{D}_{K,\mu_n}(c) = \frac{1}{n}\sum_{i=1}^{n} \min_{1\leq k\leq K} |X_i - c_k|^2 = \frac{1}{n}\sum_{i=1}^{n} |X_i|^2 + \min_{1\leq k\leq K}\left(-\frac{2}{n}\sum_{i=1}^{n}\langle X_i|c_k\rangle + |c_k|^2\right).$$

The a.s. convergence of optimal quantizers for the empirical measure has been proved in Pollard (1981). We give a first upper bound of the clustering performance by applying directly Theorem 4 and (21).

Proposition 13 *Let $K \in \mathbb{N}^*$ be the quantization level. Let $\mu \in \mathcal{P}_q(\mathbb{R}^d)$ for some $q > 2$ with card(supp(μ)) $\geq K$ and let μ_n^ω be the empirical measure of μ defined in (20). Let $x^{(n),\omega}$ be an optimal quantizer at level K of μ_n^ω. Then for any $n > K$,*

$$\mathbb{E}\,\mathcal{D}_{K,\mu}(x^{(n),\omega}) - \inf_{x\in(\mathbb{R}^d)^K} \mathcal{D}_{K,\mu}(x) \leq C_{d,q,\mu,K} \times \begin{cases} n^{-1/4} + n^{-(q-2)/2q} & \text{if } d < 4 \text{ and } q \neq 4 \\ n^{-1/4}\big(\log(1+n)\big)^{1/2} + n^{-(q-2)/2q} & \text{if } d = 4 \text{ and } q \neq 4 \\ n^{-1/d} + n^{-(q-2)/2q} & \text{if } d > 4 \text{ and } q \neq d/(d-2) \end{cases}.$$

where $C_{d,q,\mu,K}$ is a constant depending on d, q, μ and the quantization level K.

The reason why we only consider $n > K$ is that for a fixed $n \in \mathbb{N}^*$, the empirical measure μ_n defined in (20) is supported by n points, which implies that, if $n \leq K$, the optimal quantizer of μ_n at level K, viewed as a set, is in fact supp(μ_n). This makes the above bound of no interest. Following the remark after Theorem 1 in Fournier and Guillin (2015), one can remark that if the probability distribution μ has sufficiently large moments (namely if $q > 4$ when $d \leq 4$ and $q > 2d/(d-2)$ when $d > 4$), then the term $n^{-(q-2)/2q}$ is negligible and can be removed.

Proof [Proof of Proposition 13] For every $\omega \in \Omega$ and for every $n > K$, Theorem 4 implies that

$$\mathcal{D}_{K,\mu}(x^{(n),\omega}) - \inf_{x\in(\mathbb{R}^d)^K} \mathcal{D}_{K,\mu}(x) \leq 4e^*_{K,\mu}\mathcal{W}_2(\mu_n^\omega, \mu) + 4\mathcal{W}_2^2(\mu_n^\omega, \mu).$$

Thus,

$$\mathbb{E}\,\mathcal{D}_{K,\mu}(x^{(n),\omega}) - \inf_{x\in(\mathbb{R}^d)^K} \mathcal{D}_{K,\mu}(x) \leq 4e^*_{K,\mu}\mathbb{E}\,\mathcal{W}_2(\mu_n^\omega, \mu) + 4\,\mathbb{E}\,\mathcal{W}_2^2(\mu_n^\omega, \mu).$$

It follows from (21) applied with $p = 2$ that

$$\mathbb{E}\,\mathcal{W}_2^2(\mu_n^\omega, \mu) \leq C_{d,q,\mu} \times \begin{cases} n^{-1/2} + n^{-(q-2)/q} & \text{if } d < 4 \text{ and } q \neq 4 \\ n^{-1/2}\log(1+n) + n^{-(q-2)/q} & \text{if } d = 4 \text{ and } q \neq 4 \\ n^{-2/d} + n^{-(q-2)/q} & \text{if } d > 4 \text{ and } q \neq d/(d-2) \end{cases}, \tag{22}$$

where $C_{d,q,\mu} = C \cdot M_q^{2/q}(\mu)$ and C is the constant in (21). Moreover, as $\mathbb{E}\,\mathcal{W}_2(\mu_n^\omega, \mu) \leq \big(\mathbb{E}\mathcal{W}_2^2(\mu_n^\omega, \mu)\big)^{1/2}$ and $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for any $a, b \in \mathbb{R}_+$, Inequality (21) also implies

$$\mathbb{E}\,\mathcal{W}_2(\mu_n^\omega, \mu) \leq C_{d,q,\mu}^{1/2} \times \begin{cases} n^{-1/4} + n^{-(q-2)/2q} & \text{if } d < 4 \text{ and } q \neq 4 \\ n^{-1/4}\big(\log(1+n)\big)^{1/2} + n^{-(q-2)/2q} & \text{if } d = 4 \text{ and } q \neq 4 \\ n^{-1/d} + n^{-(q-2)/2q} & \text{if } d > 4 \text{ and } q \neq d/(d-2) \end{cases}.$$

Consequently,

$$\mathbb{E}\,\mathcal{D}_{K,\mu}(x^{(n),\omega}) - \inf_{x\in(\mathbb{R}^d)^K} \mathcal{D}_{K,\mu}(x) \leq 4e^*_{K,\mu}\mathbb{E}\,\mathcal{W}_2(\mu_n^\omega, \mu) + 4\,\mathbb{E}\,\mathcal{W}_2^2(\mu_n^\omega, \mu).$$

$$\leq 8(C_{d,q,\mu}^{1/2} e_{K,\mu}^{*} \vee C_{d,q,\mu})\times \begin{cases} n^{-1/4} + n^{-(q-2)/2q} & \text{if } d<4 \text{ and } q\neq 4 \\ n^{-1/4}\big(\log(1+n)\big)^{1/2} + n^{-(q-2)/2q} & \text{if } d=4 \text{ and } q\neq 4 \\ n^{-1/d} + n^{-(q-2)/2q} & \text{if } d>4 \text{ and } q\neq d/(d-2) \end{cases}. \quad (23)$$

One can conclude by setting $C_{d,q,\mu,K} \coloneqq 8(C_{d,q,\mu}^{1/2} e_{K,\mu}^{*} \vee C_{d,q,\mu})$. ■

Remark 14 *When $d \geq 4$, if $\frac{q-2}{q} > \frac{2}{d}$ i.e. $q > \frac{2d}{d-2}$, Inequality (22) can be upper bounded as follows,*

$$\begin{aligned} \mathbb{E}\,\mathcal{W}_2^2(\mu_n^\omega,\mu) &\leq 2C_{d,q,\mu} n^{-1/d} \times \begin{cases} n^{-\frac{1}{4}}\log(1+n) & \text{if } d=4 \text{ and } q\neq 4 \\ n^{-\frac{1}{d}} & \text{if } d>4 \text{ and } q\neq d/(d-2) \end{cases} \\ &\leq 2C_{d,q,\mu} K^{-1/d} \times \begin{cases} n^{-\frac{1}{4}}\log(1+n) & \text{if } d=4 \text{ and } q\neq 4 \\ n^{-\frac{1}{d}} & \text{if } d>4 \text{ and } q\neq d/(d-2) \end{cases}, \end{aligned}$$

since we consider only $n \geq K$ and if $q > \frac{2d}{d-2}$, the term $n^{-(q-2)/2q}$ becomes negligible as n grows. Consequently, (23) can be bounded by

$$\begin{aligned} &\mathbb{E}\,\mathcal{D}_{K,\mu}(x^{(n),\omega}) - \inf_{x\in(\mathbb{R}^d)^K} \mathcal{D}_{K,\mu}(x) \leq 4e_{K,\mu}^{*}\mathbb{E}\,\mathcal{W}_2(\mu_n^\omega,\mu) + 4\,\mathbb{E}\,\mathcal{W}_2^2(\mu_n^\omega,\mu). \\ &\leq 8(C_{d,q,\mu}^{1/2} e_{K,\mu}^{*} \vee 2C_{d,q,\mu}K^{-1/d})\times \\ &\begin{cases} n^{-\frac{1}{4}}\big[(\log(1+n))^{\frac{1}{2}} + \log(1+n)\big] & \text{if } d=4 \text{ and } q\neq 4 \\ 2n^{-\frac{1}{d}} & \text{if } d>4 \text{ and } q\neq d/(d-2) \end{cases}. \end{aligned} \quad (24)$$

By the non-asymptotic Zador theorem (10), one has

$$e_{K,\mu}^{*} \leq C_{d,q}(\mu)\sigma_q(\mu)K^{-1/d}$$

with $\sigma_q(\mu) = \min_{a\in\mathbb{R}^d}\left[\int_{\mathbb{R}^d}|\xi-a|^q\,\mu(d\xi)\right]^{1/q}$. Thus, Inequality (24) can be upper-bounded as follows,

$$\begin{aligned} &\mathbb{E}\,\mathcal{D}_{K,\mu}(x^{(n),\omega}) - \inf_{x\in(\mathbb{R}^d)^K} \mathcal{D}_{K,\mu}(x) \leq 4e_{K,\mu}^{*}\mathbb{E}\,\mathcal{W}_2(\mu_n^\omega,\mu) + 4\,\mathbb{E}\,\mathcal{W}_2^2(\mu_n^\omega,\mu). \\ &\leq 8K^{-1/d}\big(C_{d,q,\mu}^{1/2} C_{d,q}(\mu)\sigma_q(\mu) \vee 2C_{d,q,\mu}\big)\times \\ &\begin{cases} n^{-\frac{1}{4}}\big[(\log(1+n))^{\frac{1}{2}} + \log(1+n)\big] & \text{if } d=4 \text{ and } q\neq 4 \\ 2n^{-\frac{1}{d}} & \text{if } d>4 \text{ and } q\neq d/(d-2) \end{cases}, \end{aligned}$$

from which one can remark that the constant $C_{d,q,\mu,K}$ in Proposition 13 is roughly decreasing as $K^{-1/d}$.

A second upper bound of the clustering performance is provided in the following theorem.

Theorem 15 *Let $K \in \mathbb{N}^*$ be the quantization level. Let $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ with card(supp(μ)) $\geq K$ and let μ_n^ω be the empirical measures of μ defined in (20), generated by i.i.d observations $X_1, ..., X_n, ...$ We denote by $x^{(n),\omega} \in (\mathbb{R}^d)^K$ an optimal quantizer of μ_n^ω at level K. Then,*

(a) General upper bound of the performance.

$$\mathbb{E}\, \mathcal{D}_{K,\mu}(x^{(n),\omega}) - \inf_{x \in (\mathbb{R}^d)^K} \mathcal{D}_{K,\mu}(x) \leq \frac{2K}{\sqrt{n}}\Big[r_{2n}^2 + \rho_K(\mu)^2 + 2r_1\big(r_{2n} + \rho_K(\mu)\big)\Big],$$

where $r_n := \big\| \max_{1 \leq i \leq n} |X_i| \, \big\|_2$ and $\rho_K(\mu)$ is the maximum radius of optimal quantizers of μ, defined in (9).

(b) Asymptotic upper bound for distribution with polynomial tail. *For $p > 2$, if μ has a c-th polynomial tail with $c > d + p$, then*

$$\mathbb{E}\, \mathcal{D}_{K,\mu}(x^{(n),\omega}) - \inf_{x \in (\mathbb{R}^d)^K} \mathcal{D}_{K,\mu}(x) \leq \frac{K}{\sqrt{n}}\Big[C_{\mu,p}\, n^{2/p} + 6K^{\frac{2(p+d)}{d(c-p-d)}\gamma_K}\Big],$$

where $C_{\mu,p}$ is a constant depending μ, p and $\lim_K \gamma_K = 1$.

(c) Asymptotic upper bound for distribution with hyper-exponential tail. *Recall that μ has a hyper-exponential tail if $\mu = f \cdot \lambda_d$ and there exists $\tau > 0, \kappa, \vartheta > 0, c > -d$ and $A > 0$ such that $\forall \xi \in \mathbb{R}^d, |\xi| \geq A \Rightarrow f(\xi) = \tau\,|\xi|^c\, e^{-\vartheta|\xi|^\kappa}$. If $\kappa \geq 2$, we can obtain a more precise upper bound of the performance*

$$\mathbb{E}\big[\mathcal{D}_{K,\mu}(x^{(n),\omega}) - \inf_{x \in (\mathbb{R}^d)^K} \mathcal{D}_{K,\mu}(x)\big] \leq C_{\vartheta,\kappa,\mu} \cdot \frac{K}{\sqrt{n}}\Big[1 + (\log n)^{2/\kappa} + \gamma_K (\log K)^{2/\kappa}\big(1 + \frac{2}{d}\big)^{2/\kappa}\Big],$$

where $C_{\vartheta,\kappa,\mu}$ is a constant depending ϑ, κ, μ and $\limsup_K \gamma_K = 1$.
In particular, if $\mu = \mathcal{N}(m, \Sigma)$, the multidimensional normal distribution, we have

$$\mathbb{E}\big[\mathcal{D}_{K,\mu}(x^{(n),\omega}) - \inf_{x \in (\mathbb{R}^d)^K} \mathcal{D}_{K,\mu}(x)\big] \leq C_\mu \cdot \frac{K}{\sqrt{n}}\Big[1 + \log n + \gamma_K \cdot (\log K)\big(1 + \frac{2}{d}\big)\Big],$$

where $\limsup_K \gamma_K = 1$ and $C_\mu = 24 \cdot \big(1 \vee \log 2\mathbb{E}e^{|X|^2/4}\big)$ where X is a random variable with distribution μ. Moreover, when $\mu = \mathcal{N}(0, \mathbf{I}_d)$, $C_\mu = 24(1 + \frac{d}{2}) \cdot \log 2$.

The proof of Theorem 15 relies on the Rademacher process theory. A Rademacher sequence $(\sigma_i)_{i \in \{1,...,n\}}$ is a sequence of i.i.d random variables with a symmetric $\{\pm 1\}$-valued Bernoulli distribution, independent of $(X_1, ..., X_n)$ and we define the Rademacher process $\mathcal{R}_n(f), f \in \mathcal{F}$ by $\mathcal{R}_n(f) := \frac{1}{n}\sum_{i=1}^n \sigma_i f(X_i)$. Remark that the Rademacher process $\mathcal{R}_n(f)$ depends on the sample $\{X_1, ..., X_n\}$ of the probability measure μ.

Theorem (Symmetrization inequalites) *For any class $\mathcal{F}$ of μ-integrable functions, we have*

$$\mathbb{E}\, \|\mu_n - \mu\|_{\mathcal{F}} \leq 2\mathbb{E}\, \|\mathcal{R}_n\|_{\mathcal{F}}\,,$$

where for a probability distribution ν, $\|\nu\|_{\mathcal{F}} := \sup_{f \in \mathcal{F}} |\nu(f)| := \sup_{f \in \mathcal{F}} \big|\int_{\mathbb{R}^d} f d\nu\big|$ and $\|\mathcal{R}_n\|_{\mathcal{F}} := \sup_{f \in \mathcal{F}} |\mathcal{R}_n(f)|$.

For the proof of the above theorem, we refer to Koltchinskii (2011, Theorem 2.1). Another more detailed reference is Van Der Vaart and Wellner (1996, Lemma 2.3.1). We will also introduce the *Contraction principle* in the following theorem and we refer to Boucheron et al. (2013, Theorem 11.6) for the proof.

Theorem (Contraction principle) *Let $x_1, ..., x_n$ be vectors whose real-valued components are indexed by $\mathcal{T}$, that is, $x_i = (x_{i,s})_{s\in\mathcal{T}}$. For each $i = 1, ..., n$, let $\varphi_i : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function such that $\varphi_i(0) = 0$. Let $\sigma_1, ..., \sigma_n$ be independent Rademacher random variables and let $c_L = \max_{1\le i\le n} \sup_{\substack{x,y\in\mathbb{R}\\ x\neq y}} \left|\frac{\varphi_i(x)-\varphi_i(y)}{x-y}\right|$ be the uniform Lipschitz constant of the function φ_i. Then*

$$\mathbb{E}\Big[\sup_{s\in\mathcal{T}} \sum_{i=1}^{n} \sigma_i\varphi_i(x_{i,s})\Big] \le c_L \cdot \mathbb{E}\Big[\sup_{s\in\mathcal{T}} \sum_{i=1}^{n} \sigma_i x_{i,s}\Big]. \tag{25}$$

Remark that, if we consider random variables $(Y_{1,s}, ..., Y_{n,s})_{s\in\mathcal{T}}$ independent of $(\sigma_1, ..., \sigma_n)$ and for all $s \in \mathcal{T}$ and $i \in \{1, ..., n\}$, $Y_{i,s}$ is valued in $\mathbb{R}$, then (25) implies that

$$\begin{aligned}\mathbb{E}\Big[\sup_{s\in\mathcal{T}} \sum_{i=1}^{n}\sigma_i\varphi_i(Y_{i,s})\Big] &= \mathbb{E}\Big\{\mathbb{E}\Big[\sup_{s\in\mathcal{T}} \sum_{i=1}^{n} \sigma_i\varphi_i(Y_{i,s}) \mid (Y_{1,s}, ..., Y_{n,s})_{s\in\mathcal{T}}\Big]\Big\}\\ &\le c_L \cdot \mathbb{E}\Big\{\mathbb{E}\Big[\sup_{s\in\mathcal{T}} \sum_{i=1}^{n} \sigma_i Y_{i,s} \mid (Y_{1,s}, ..., Y_{n,s})_{s\in\mathcal{T}}\Big]\Big\} \le c_L \cdot \mathbb{E}\Big[\sup_{s\in\mathcal{T}} \sum_{i=1}^{n} \sigma_i Y_{i,s}\Big]. \end{aligned}\tag{26}$$

The proof of Theorem 15 is inspired by that of Theorem 2.1 in Biau et al. (2008).

Proof [Proof of Theorem 15] (a) In order to simplify the notation, we will denote by $\mathcal{D}$ (*respectively* $\mathcal{D}_n$) instead of $\mathcal{D}_{K,\mu}$ (*resp.* $\mathcal{D}_{K,\mu_n}$) the distortion function of μ (*resp.* μ_n). For any $c = (c_1, ..., c_K) \in (\mathbb{R}^d)^K$, note that the distortion function $\mathcal{D}(c)$ of μ can be written as

$$\mathcal{D}(c) = \mathbb{E}\big[\min_{1\le k\le K} |X - c_k|^2\big] = \mathbb{E}\big[|X|^2 + \min_{1\le k\le K}(-2\langle X|c_k\rangle + |c_k|^2)\big].$$

We define $\overline{\mathcal{D}}(c) \coloneqq \min_{1\le k\le K}\big(-2\langle X|c_k\rangle + |c_k|^2\big)$. Similarly, for the distortion function $\mathcal{D}_n$ of the empirical measure μ_n,

$$\mathcal{D}_n(c) = \frac{1}{n}\sum_{i=1}^{n} \min_{1\le k\le K} |X_i - c_k|^2 = \frac{1}{n}\sum_{i=1}^{n} |X_i|^2 + \min_{1\le k\le K}\Big(-\frac{2}{n}\sum_{i=1}^{n}\langle X_i|c_k\rangle + |c_k|^2\Big),$$

we define $\overline{\mathcal{D}}_n(c) \coloneqq \min_{1\le k\le K}\big(-\frac{2}{n}\sum_{i=1}^{n}\langle X_i|c_k\rangle + |c_k|^2\big)$. We will drop ω in $x^{(n),\omega}$ to alleviate the notation throughout the proof. Let $x \in \operatorname{argmin} \mathcal{D}_{K,\mu}$. It follows that

$$\begin{aligned}\mathbb{E}\big[\mathcal{D}(x^{(n)}) - \mathcal{D}(x)\big] &= \mathbb{E}\big[\overline{\mathcal{D}}(x^{(n)}) - \overline{\mathcal{D}}(x)\big] = \mathbb{E}\big[\overline{\mathcal{D}}(x^{(n)}) - \overline{\mathcal{D}}_n(x^{(n)})\big] + \mathbb{E}\big[\overline{\mathcal{D}}_n(x^{(n)}) - \overline{\mathcal{D}}(x)\big]\\ &\le \mathbb{E}\big[\overline{\mathcal{D}}(x^{(n)}) - \overline{\mathcal{D}}_n(x^{(n)})\big] + \mathbb{E}\big[\overline{\mathcal{D}}_n(x) - \overline{\mathcal{D}}(x)\big].\end{aligned}$$

Define for $\eta, x \in \mathbb{R}^d$, $f_\eta(x) \coloneqq -2\langle\eta|x\rangle + |\eta|^2$.

Part (i): Upper bound of $\mathbb{E}[\overline{\mathcal{D}}(x^{(n)}) - \overline{\mathcal{D}}_n(x^{(n)})]$. Let $R_n(\omega) \coloneqq \max_{1\le i\le n}|X_i(\omega)|$. Remark that for every $\omega\in\Omega$, $R_n(\omega)$ is invariant with the respect to all permutations of the components of $(X_1,...,X_n)$. Let B_R denote the ball centred at 0 with radius R. Then, owing to Proposition 2-(iii), $x^{(n)} = (x_1^{(n)},...,x_K^{(n)}) \in B_{R_n}^K$. Hence,

$$\begin{aligned}\mathbb{E}\,[\overline{\mathcal{D}}(x^{(n)}) - \overline{\mathcal{D}}_n(x^{(n)})] &\le \mathbb{E}\sup_{c\in B_{R_n}^K}\big(\overline{\mathcal{D}}(c) - \overline{\mathcal{D}}_n(c)\big)\\ &= \mathbb{E}\,\big[\sup_{c\in B_{R_n}^K}\big(\mathbb{E}\min_{1\le k\le K} f_{c_k}(X) - \frac{1}{n}\sum_{i=1}^n \min_{1\le k\le K} f_{c_k}(X_i)\big)\big]\\ &= \mathbb{E}\,\Big[\sup_{c\in B_{R_n}^K}\mathbb{E}\big[\frac{1}{n}\sum_{i=1}^n \min_{1\le k\le K} f_{c_k}(X_i') - \frac{1}{n}\sum_{i=1}^n \min_{1\le k\le K} f_{c_k}(X_i)\big| X_1,...,X_n\big]\Big], \qquad (27)\end{aligned}$$

where $X_1',...,X_n'$ are i.i.d random variable with the distribution μ, independent of $(X_1,...,X_n)$. Let $R_{2n} \coloneqq \max_{1\le i\le n}|X_i|\vee|X_i'|$, then (27) becomes

$$\begin{aligned}\mathbb{E}\,[\overline{\mathcal{D}}(x^{(n)}) - \overline{\mathcal{D}}_n(x^{(n)})] &\le \mathbb{E}\,\Big[\sup_{c\in B_{R_{2n}}^K}\mathbb{E}\big[\frac{1}{n}\sum_{i=1}^n \min_{1\le k\le K} f_{c_k}(X_i') - \frac{1}{n}\sum_{i=1}^n \min_{1\le k\le K} f_{c_k}(X_i)\big| X_1,...,X_n\big]\Big]\\ &\le \mathbb{E}\,\Big[\mathbb{E}\big[\sup_{c\in B_{R_{2n}}^K}\big(\frac{1}{n}\sum_{i=1}^n \min_{1\le k\le K} f_{c_k}(X_i') - \frac{1}{n}\sum_{i=1}^n \min_{1\le k\le K} f_{c_k}(X_i)\big)\big| X_1,...,X_n\big]\Big]\\ &= \mathbb{E}\big[\sup_{c\in B_{R_{2n}}^K}\frac{1}{n}\sum_{i=1}^n\big(\min_{1\le k\le K} f_{c_k}(X_i') - \min_{1\le k\le K} f_{c_k}(X_i)\big)\big].\end{aligned}$$

The distribution of $(X_1,...,X_n,X_1',...,X_n')$ and that of R_{2n} are invariant with the respect to all permutation of the components in $(X_1,...,X_n,X_1',...,X_n')$. Hence,

$$\begin{aligned}\mathbb{E}\,[\overline{\mathcal{D}}(x^{(n)}) - \overline{\mathcal{D}}_n(x^{(n)})] &= \mathbb{E}\big[\sup_{c\in B_{R_{2n}}^K}\frac{1}{n}\sum_{i=1}^n\sigma_i\big(\min_{1\le k\le K} f_{c_k}(X_i') - \min_{1\le k\le K} f_{c_k}(X_i)\big)\big]\\ &\le \mathbb{E}\big[\sup_{c\in B_{R_{2n}}^K}\frac{1}{n}\sum_{i=1}^n\sigma_i\min_{1\le k\le K} f_{c_k}(X_i')\big] + \mathbb{E}\big[\sup_{c\in B_{R_{2n}^K}}\frac{1}{n}\sum_{i=1}^n\sigma_i\min_{1\le k\le K} f_{c_k}(X_i)\big]\\ &= 2\mathbb{E}\big[\sup_{c\in B_{R_{2n}}^K}\frac{1}{n}\sum_{i=1}^n\sigma_i\min_{1\le k\le K} f_{c_k}(X_i)\big]. \qquad (28)\end{aligned}$$

In the second line of (28), we can change the sign before the second term since $-\sigma_i$ has the same distribution of σ_i, and we will continue to use this property throughout the proof. Let $S_K = \mathbb{E}\Big[\sup_{c\in B_{R_{2n}}^K}\frac{1}{n}\sum_{i=1}^n\sigma_i\min_{1\le k\le K} f_{c_k}(X_i)\Big]$ and we provide an upper bound for S_K by induction on K in what follows.

► For $K=1$,

$$S_1 = \mathbb{E}\big[\sup_{c\in B_{R_{2n}}}\frac{1}{n}\sum_{i=1}^n\sigma_i\min_{1\le k\le K} f_c(X_i)\big] = \mathbb{E}\big[\sup_{c\in B_{R_{2n}}}\frac{1}{n}\sum_{i=1}^n\sigma_i\big(-2\langle c|X_i\rangle + |c|^2\big)\big]$$

$$\begin{aligned}
&\leq 2\,\mathbb{E}\big[\sup_{c\in B_{R_{2n}}}\frac{1}{n}\sum_{i=1}^{n}\sigma_i\langle c|X_i\rangle\big]+\mathbb{E}\big[\sup_{c\in B_{R_{2n}}}\frac{1}{n}\sum_{i=1}^{n}\sigma_i\,|c|^2\,\big]\\
&\leq \frac{2}{n}\mathbb{E}\Big[\sup_{c\in B_{R_{2n}}}\langle c|\sum_{i=1}^{n}\sigma_iX_i\rangle\Big]+\frac{1}{n}\mathbb{E}\Big[\Big|\sum_{i=1}^{n}\sigma_i\Big|\cdot|R_{2n}|^2\,\Big]\\
&\leq \frac{2}{n}\mathbb{E}\Big[\sup_{c\in B_{R_{2n}}}\Big|\sum_{i=1}^{n}\sigma_iX_i\Big|\cdot|c|\,\Big]+\frac{1}{n}\mathbb{E}\Big|\sum_{i=1}^{n}\sigma_i\Big|\cdot\mathbb{E}\,|R_{2n}|^2\\
&\quad\text{(by Cauchy-Schwarz inequality and independence of } \sigma_i \text{ and } X_i)\\
&\leq \frac{2}{n}\Big\|\sum_{i=1}^{n}\sigma_iX_i\Big\|_2\cdot\|R_{2n}\|_2+\frac{1}{n}\Big\|\sum_{i=1}^{n}\sigma_i\Big\|_2^2\cdot\|R_{2n}\|_2^2\\
&\leq \frac{2}{n}\sqrt{n}\,\|X_1\|_2\cdot\|R_{2n}\|_2+\frac{1}{\sqrt{n}}\,\|R_{2n}\|_2^2\leq\frac{\|R_{2n}\|_2}{\sqrt{n}}\big(2\,\|X_1\|_2+\|R_{2n}\|_2\,\big).
\end{aligned}\tag{29}$$

The first inequality of the last line of (29) follows from $\mathbb{E}\,|\sum_{i=1}^{n}\sigma_iX_i|^2=\mathbb{E}\sum_{i=1}^{n}\sigma_i^2X_i^2=n\mathbb{E}X_1^2$ since the $(\sigma_1,...,\sigma_n)$ is independent of $(X_1,...,X_n)$ and $\mathbb{E}\,\sigma_i=0$. For $n\in\mathbb{N}^*$, define $r_n:=\|\max_{1\leq i\leq n}|Y_i|\|_2$, where $Y_1,...,Y_n$ are i.i.d random variables with probability distribution μ. Hence, $r_{2n}=\|R_{2n}\|_2$, since $(Y_1,...,Y_{2n})$ has the same distribution as $(X_1,...,X_n,X_1',...,X_n')$. Therefore,

$$S_1\leq\frac{r_{2n}}{\sqrt{n}}\big(2\,\|X_1\|_2+r_{2n}\big).$$

► For $K=2$,

$$\begin{aligned}
S_2&=\mathbb{E}\big[\sup_{c=(c_1,c_2)\in B^2_{R_{2n}}}\frac{1}{n}\sum_{i=1}^{n}\sigma_i\big(f_{c_1}(X_i)\wedge f_{c_2}(X_i)\big)\big]\\
&=\frac{1}{2}\mathbb{E}\Big[\sup_{c\in B^2_{R_{2n}}}\frac{1}{n}\sum_{i=1}^{n}\sigma_i\big(f_{c_1}(X_i)+f_{c_2}(X_i)-|f_{c_1}(X_i)-f_{c_2}(X_i)|\,\big)\Big]\big(\text{as } a\wedge b=\frac{a+b}{2}-\frac{|a-b|}{2}\big)\\
&\leq\frac{1}{2}\Big\{\mathbb{E}\big[\sup_{c\in B^2_{R_{2n}}}\frac{1}{n}\sum_{i=1}^{n}\sigma_i\big(f_{c_1}(X_i)+f_{c_2}(X_i)\big)\big]+\mathbb{E}\big[\sup_{c\in B^2_{R_{2n}}}\frac{1}{n}\sum_{i=1}^{n}\sigma_i\,|f_{c_1}(X_i)-f_{c_2}(X_i)|\,\big]\Big\}\\
&\leq\frac{1}{2}\Big\{2S_1+\mathbb{E}\big[\sup_{c\in B^2_{R_{2n}}}\frac{1}{n}\sum_{i=1}^{n}\sigma_i\big(f_{c_1}(X_i)-f_{c_2}(X_i)\big)\big]\Big\}\quad\big(\text{by (26)}\big)\\
&\leq\frac{1}{2}\Big\{2S_1+\mathbb{E}\big[\sup_{c_1\in B_{R_{2n}}}\frac{1}{n}\sum_{i=1}^{n}\sigma_if_{c_1}(X_i)\big]+\mathbb{E}\big[\sup_{c_2\in B_{R_{2n}}}\frac{1}{n}\sum_{i=1}^{n}\sigma_if_{c_2}(X_i)\big]\Big\}\leq 2S_1.
\end{aligned}$$

► Next, we will show by induction that $S_K\leq KS_1$ for every $K\in\mathbb{N}^*$. Assume that $S_K\leq KS_1$, for $K+1$,

$$S_{K+1}=\mathbb{E}\big[\sup_{c\in B^{K+1}_{R_{2n}}}\frac{1}{n}\sum_{i=1}^{n}\sigma_i\min_{1\leq k\leq K+1}f_{c_k}(X_i)\big]$$

$$
\begin{aligned}
&= \mathbb{E}\Big[\sup_{c\in B_{R_{2n}}^{K+1}} \frac{1}{n}\sum_{i=1}^{n}\sigma_i\big(\min_{1\le k\le K} f_{c_k}(X_i)\wedge f_{c_{K+1}}(X_i)\big)\Big]\\
&\le \frac{1}{2}\mathbb{E}\Big\{\sup_{c\in B_{R_{2n}}^{K+1}} \frac{1}{n}\sum_{i=1}^{n}\sigma_i\Big[\big(\min_{1\le k\le K} f_{c_k}(X_i)+f_{c_{K+1}}(X_i)\big)-\Big|\min_{1\le k\le K} f_{c_k}(X_i)-f_{c_{K+1}}(X_i)\Big|\Big]\Big\}\\
&\le \frac{1}{2}\mathbb{E}\Big\{\sup_{c\in B_{R_{2n}}^{K+1}} \frac{1}{n}\sum_{i=1}^{n}\sigma_i\big(\min_{1\le k\le K} f_{c_k}(X_i)+f_{c_{K+1}}(X_i)\big)\\
&\qquad+\sup_{c\in B_{R_{2n}}^{K+1}} \frac{1}{n}\sum_{i=1}^{n}\sigma_i\Big|\min_{1\le k\le K} f_{c_k}(X_i)-f_{c_{K+1}}(X_i)\Big|\Big\}\\
&\le \frac{1}{2}(S_K+S_1+S_K+S_1)\le S_K+S_1\le (K+1)S_1.
\end{aligned}
$$

Hence,

$$
\mathbb{E}\,[\overline{\mathcal{D}}(x^{(n)})-\overline{\mathcal{D}}_n(x^{(n)})]\le 2S_K\le 2KS_1\le \frac{2K\cdot r_{2n}}{\sqrt{n}}\big(2\left\|X_1\right\|_2+r_{2n}\big).
$$

Part (ii): Upper bound of $\mathbb{E}\,[\overline{\mathcal{D}}_n(x)-\overline{\mathcal{D}}(x)]$. As $x=(x_1,...,x_K)$ is an optimal quantizer of μ, we have $\max_{1\le k\le K}|x_k|\le\rho_K(\mu)$ owing to the definition of $\rho_K(\mu)$ in (9). Consequently,

$$
\mathbb{E}\big[\overline{\mathcal{D}}_n(x)-\overline{\mathcal{D}}(x)\big]\le\mathbb{E}\sup_{c\in B_{\rho_K(\mu)}^{K}}\big[\overline{\mathcal{D}}_n(c)-\overline{\mathcal{D}}(c)\big]
$$

By the same reasoning of Part (I), we have $\mathbb{E}\big[\overline{\mathcal{D}}_n(x)-\overline{\mathcal{D}}(x)\big]\le\frac{2K}{\sqrt{n}}\rho_K(\mu)\big(2\left\|X_1\right\|_2+\rho_K(\mu)\big)$. Hence

$$
\begin{aligned}
\mathbb{E}\big[\mathcal{D}(x^{(n)})-\mathcal{D}(x)\big]&\le\frac{2K}{\sqrt{n}}r_{2n}\big(2\left\|X_1\right\|_2+r_{2n}\big)+\frac{2K}{\sqrt{n}}\rho_K(\mu)\big(2\left\|X_1\right\|_2+\rho_K(\mu)\big)\\
&\le\frac{2K}{\sqrt{n}}\Big[r_{2n}^2+\rho_K^2(\mu)+2r_1\big(r_{2n}+\rho_K(\mu)\big)\Big].
\end{aligned}\tag{30}
$$

The proof of (b) and (c) is postponed in Appendix E. ■

Appendix A: Proof of Pollard's Theorem

Proof Since the quantization level K is fixed, in this proof, we drop the subscript K of the distortion function and denote by $\mathcal{D}_n$ (*respectively,* $\mathcal{D}_\infty$) the distortion function of μ_n (*resp.* μ_∞).

We know $x^{(n)}\in\text{argmin }\mathcal{D}_n$ owing to Proposition 2, that is, for all $y\in(y_1,...,y_K)\in(\mathbb{R}^d)^K$, we have $\mathcal{D}_n(x^{(n)})\le\mathcal{D}_n(y)$. For every fixed $y=(y_1,...,y_K)$, we have $\mathcal{D}_n(y)\to\mathcal{D}_\infty(y)$ after (13) so that

$$
\limsup_n\ \mathcal{D}_n(x^{(n)})\le\inf_{y\in(\mathbb{R}^d)^K}\mathcal{D}_\infty(y).\tag{31}
$$

Assume that there exists an index set $\mathcal{I}\subset\{1,...,K\}$ and $\mathcal{I}^c\neq\varnothing$ such that $(x_i^{(n)})_{i\in\mathcal{I},n\ge1}$ is bounded and $(x_i^{(n)})_{i\in\mathcal{I}^c,n\ge1}$ is not bounded. Then there exists a subsequence $\psi(n)$ of n

such that

$$\begin{cases} x_i^{\psi(n)} \to \widetilde{x}_i^{(\infty)}, & i \in \mathcal{I}, \\ \left|x_i^{\psi(n)}\right| \to +\infty, & i \in \mathcal{I}^c. \end{cases}$$

After (13), we have $\mathcal{D}_{\psi(n)}(x^{(\psi(n))})^{1/2} \geq \mathcal{D}_{\infty}(x^{(\psi(n))})^{1/2} - \mathcal{W}_2(\mu_{\psi(n)}, \mu_{\infty})$. Hence,

$$\liminf_n \mathcal{D}_{\psi(n)}(x^{(\psi(n))})^{1/2} \geq \liminf_n \mathcal{D}_{\infty}(x^{(\psi(n))})^{1/2}$$

so that

$$\begin{aligned} \liminf_n \ \mathcal{D}_{\psi(n)}(x^{(\psi(n))})^{1/2} \geq & \liminf_n \mathcal{D}_{\infty}(x^{(\psi(n))})^{1/2} \\ = & \big[\liminf_n \int \min_{i\in\{1,\dots,K\}} \left|x_i^{(\psi(n))} - \xi\right|^2 \mu_\infty(d\xi)\big]^{1/2} \\ \geq & \big[\int \liminf_n \min_{i\in\{1,\dots,K\}} \left|x_i^{(\psi(n))} - \xi\right|^2 \mu_\infty(d\xi)\big]^{1/2} \\ = & \big[\int \min_{i\in\mathcal{I}} \left|x_i^{(\infty)} - \xi\right|^2 \mu_\infty(d\xi)\big]^{1/2}, \end{aligned} \tag{32}$$

where we used Fatou's Lemma in the third line. Thus, (31) and (32) imply that

$$\int \min_{i\in\mathcal{I}} \left|x_i^{(\infty)} - \xi\right|^2 \mu_\infty(d\xi) \leq \inf_{y\in(\mathbb{R}^d)^K} \mathcal{D}_\infty(y). \tag{33}$$

This implies that $\mathcal{I} = \{1, ..., K\}$ after Proposition 2 (otherwise, (33) implies that $e^{|\mathcal{I}|,*}(\mu_\infty) \leq e^{K,*}(\mu_\infty)$ with $|\mathcal{I}| < K$, which is contradictory to Proposition 2-(i)). Therefore, $(x^{(n)})$ is bounded and any limiting point $x^{(\infty)} \in \mathrm{argmin}_{x\in(\mathbb{R}^d)^K} \mathcal{D}_\infty(x)$. ■

Appendix B: Proof of Proposition 2 - (iii)

We define the *open Voronoï cell* generated by x_i with respect to the Euclidean norm $|\cdot|$ by

$$V_{x_i}^o(x) = \big\{\xi \in \mathbb{R}^d \ \big| \ |\xi - x_i| < \min_{1\leq j\leq K, j\neq i} |\xi - x_j| \ \big\}.$$

It follows from Graf and Luschgy (2000, Proposition 1.3) that $\mathrm{int}V_{x_i}(x) = V_{x_i}^o(x)$, where $\mathrm{int}A$ denotes the interior of a set A. Moreover, if we denote by λ_d the Lebesgue measure on $\mathbb{R}^d$, we have $\lambda_d\big(\partial V_{x_i}(x)\big) = 0$, where ∂A denotes the boundary of A (see Graf and Luschgy, 2000, Theorem 1.5). If $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and x^* is an optimal quantizer of μ, even if μ is not absolutely continuous with the respect of λ_d, we have $\mu\big(\partial V_{x_i}(x^*)\big) = 0$ for all $i \in \{1, ..., K\}$ (see Graf and Luschgy, 2000, Theorem 4.2).

Proof Assume that there exists an $x^* = (x_1^*, ..., x_K^*) \in G_K(\mu)$ in which there exists $k \in \{1, ..., K\}$ such that $x_k^* \notin \mathcal{H}_\mu$.

Case (I): $\mu\big(V_{x_k^*}^o(\Gamma^*) \cap \mathrm{supp}(\mu)\big) = 0$. The distortion function can be written as

$$\mathcal{D}_{K,\mu}(x^*) = \sum_{i=1}^K \int_{C_{x_i}(x)} |\xi - x_i^*|^2 \mu(d\xi) = \sum_{i=1}^K \int_{V_{x_i}^o(x)} |\xi - x_i^*|^2 \mu(d\xi)$$

(since x^* is optimal and $|\cdot|$ is Euclidean, $\mu\big(\partial V_{x_i}(\Gamma^*)\big) = 0$ and $\text{int}V_{x_i}(\Gamma) = V^o_{x_i}(\Gamma)$)

$$= \sum_{i=1, i\neq k}^{K} \int_{V^o_{x_i}(x)} |\xi - x_i^*|^2 \,\mu(d\xi) = \mathcal{D}_{K,\mu}(\widetilde{x}),$$

where $\widetilde{x} = (x_1^*, ..., x_{k-1}^*, x_{k+1}^*, ..., x_K^*)$. Therefore, $\widetilde{\Gamma} = \{x_1^*, ..., x_{k-1}^*, x_{k+1}^*, ..., x_K^*\}$ is also a K-level optimal quantizer with card$(\widetilde{\Gamma}) < K$, contradictory to Proposition 2 - (i).

Case (II): $\mu\big(V^o_{x_k^*}(\Gamma^*) \cap \text{supp}(\mu)\big) > 0$. Since $x_k^* \neq \mathcal{H}_\mu$, there exists a hyperplane H strictly separating x_k^* and $\mathcal{H}_\mu$. Let $\hat{x}_k^*$ be the orthogonal projection of x_k^* on H. For any $z \in \mathcal{H}_\mu$, let b denote the point in the segment joining z and x_k^* which lies on H, then $\langle b - \hat{x}_k^* | x_k^* - \hat{x}_k^* \rangle = 0$. Hence,

$$|x_k^* - b|^2 = |\hat{x}_k^* - b|^2 + |x_k^* - \hat{x}_k^*|^2 > |\hat{x}_k^* - b|^2 .$$

Therefore, $|z - \hat{x}_k^*| \leq |z - b| + |b - \hat{x}_k^*| < |z - b| + |x_k^* - b| = |z - x_k^*|$.

Let $B(x, r)$ denote the ball on $\mathbb{R}^d$ centered at x with radius r. Since $\mu\big(V^o_{x_k^*}(\Gamma^*) \cap \text{supp}(\mu)\big) > 0$, there exists $\alpha \in V^o_{x_k^*}(\Gamma^*) \cap \text{supp}(\mu)$ such that $\exists\, r \geq 0$, $\mu\big(B(\alpha, r)\big) > 0$ (when $r = 0$, $B(\alpha, r) = \{r\}$). Moreover,

$$\forall \beta \in B(\alpha, r), \quad |\beta - \hat{x}_k^*| < |\beta - x_k^*| < \min_{i \neq k} |\beta - \hat{x}_i^*| . \tag{34}$$

Let $\hat{x} \coloneqq (x_1^*, ..., x_{k-1}^*, \hat{x}_k^*, x_{k+1}^*, ..., x_K^*)$, (34) implies $\mathcal{D}_{K,\mu}(\hat{x}) < \mathcal{D}_{K,\mu}(x^*)$. This is contradictory with the fact that x^* is an optimal quantizer. Hence, $x^* \in \mathcal{H}_\mu$. ■

Appendix C: Proof of Proposition 8

We use Lemma 11 in Fort and Pagès (1995) to compute the Hessian matrix $H_{\mathcal{D}_{K,\mu}}$ of $\mathcal{D}_{K,\mu}$.

Lemma 16 *(Fort and Pagès, 1995, Lemma 11) Let φ be a countinous $\mathbb{R}$-valued function defined on $[0,1]^d$. For every $x \in D_K \coloneqq \big\{y \in \big([0,1]^d\big)^K \mid y_i \neq y_j$ if $i \neq j\big\}$, let $\Phi_i(x) \coloneqq \int_{V_i(x)} \varphi(\omega) d\omega$. Then Φ_i is continuously differentiable on D_K and*

$$\forall i \neq j,\ \frac{\partial \Phi_i}{\partial x_j}(x) = \int_{V_i(x) \cap V_j(x)} \varphi(\xi) \big\{ \frac{1}{2} \overrightarrow{n}_x^{ij} + \frac{1}{|x_j - x_i|} \times (\frac{x_i + x_j}{2} - \xi) \big\} \lambda_x^{ij}(d\xi)$$

$$\textit{and } \frac{\partial \Phi_i}{\partial x_i}(x) = - \sum_{1 \leq j \leq K, j \neq i} \frac{\partial \Phi_j}{\partial x_i}(x),$$

where $\overrightarrow{n}_x^{ij} \coloneqq \frac{x_j - x_i}{|x_j - x_i|}$,

$$M_{ij}^x \coloneqq \Big\{ u \in \mathbb{R}^d \mid \langle u - \frac{x_i + x_j}{2} \mid x_i - x_j \rangle = 0 \Big\} \tag{35}$$

and $\lambda_x^{ij}(d\xi)$ denotes the Lebesgue measure on the affine hyperplane M_{ij}^x.

Note that one can simplify the result of Lemma 16 as follows,

$$\begin{aligned}\forall i\neq j,\quad \frac{\partial \Phi_i}{\partial x_j}(x) &= \int_{V_i(x)\cap V_j(x)} \varphi(\xi)\Big\{\frac{1}{2}\frac{x_j-x_i}{|x_j-x_i|}+\frac{1}{|x_j-x_i|}\big(\frac{x_i+x_j}{2}-\xi\big)\Big\}\lambda_x^{ij}(d\xi)\\ &= \int_{V_i(x)\cap V_j(x)} \varphi(\xi)\frac{1}{|x_j-x_i|}\Big\{\frac{x_j-x_i}{2}+\frac{x_i+x_j}{2}-\xi\Big\}\lambda_x^{ij}(d\xi)\\ &= \int_{V_i(x)\cap V_j(x)} \varphi(\xi)\frac{1}{|x_j-x_i|}(x_j-\xi)\lambda_x^{ij}(d\xi). \end{aligned} \tag{36}$$

Proof [Proof of Proposition 8] (i) Set $\varphi^{i,M}(\xi)=(x_i-\xi)f(\xi)\chi_M(\xi)$ with

$$\chi_M(\xi) := \begin{cases} 1 & |\xi|\leq M\\ M+1-|\xi| & M<|\xi|\leq M+1\\ 0 & |\xi|>M+1\end{cases}.$$

Set $\Phi_i^M(x)=\int_{V_i(x)}\varphi^{i,M}(\xi)d\xi$ and $\Phi_i(x)=\int_{V_i(x)}(x_i-\xi)f(\xi)d\xi$ for $i=1,...,K$. Then (15) implies that $\frac{\partial \mathcal{D}_{K,\mu}}{\partial x_i}=2\Phi_i,\ i=1,...,K$.

For $j=1,...,K$ and $j\neq i$, it follows from (36) that

$$\frac{\partial \Phi_i^M}{\partial x_j}(x)=\int_{V_i(x)\cap V_j(x)}(x_i-\xi)\otimes(x_j-\xi)\cdot\frac{1}{|x_j-x_i|}f(\xi)\chi_M(\xi)\lambda_x^{ij}(d\xi), \tag{37}$$

and for $i=1,...,K$,

$$\frac{\partial \Phi_i^M}{\partial x_i}(x)=\Big[\Big(\int_{V_i(\xi)}f(\xi)\chi_M(\xi)d\xi\Big)\mathrm{I}_d-\sum_{\substack{i\neq j\\1\leq j\leq K}}\int_{V_i(x)\cap V_j(x)}(x_i-\xi)\otimes(x_i-\xi)\cdot\frac{1}{|x_j-x_i|}f(\xi)\chi_M(\xi)\lambda_x^{ij}(d\xi)\Big], \tag{38}$$

where in (37) and (38), $u\otimes v:=[u^iv^j]_{1\leq i,j\leq d}$ for any two vectors $u=(u^1,...,u^d)$ and $v=(v^1,...,v^d)$ in $\mathbb{R}^d$.

We prove now the differentiability of Φ_i in three steps.

► *Step 1*: We prove in this part that for every $x\in F_K$,

$$h_{ij}(x):=\int_{V_i(x)\cap V_j(x)}(x_i-\xi)\otimes(x_j-\xi)\cdot\frac{1}{|x_j-x_i|}f(\xi)\lambda_x^{ij}(d\xi)<+\infty.$$

If $V_i(x)\cap V_j(x)=\varnothing$, it is obvious that $h_{ij}(x)=0<+\infty$. Now we assume that $V_i(x)\cap V_j(x)\neq\varnothing$. Without loss of generality, we assume that $V_1(x)\cap V_2(x)=\varnothing$ and we prove in the following h_{12} is well defined i.e. $(h_{12}(x)\in\mathbb{R}$.

Let

$$\alpha(x,\xi):=(x_1-\xi)\otimes(x_2-\xi)\cdot\frac{1}{|x_2-x_1|}f(\xi). \tag{39}$$

Then

$$h_{12}(x)=\int_{V_1(x)\cap V_2(x)}\alpha(x,\xi)\lambda_x^{12}(d\xi).$$

Let $(e_1,...,e_d)$ denote the canonical basis of $\mathbb{R}^d$. Set $u^x=\frac{x_1-x_2}{|x_1-x_2|}$. As $x_1\neq x_2$, there exists at least one $i_0\in\{1,...,d\}$ s.t. $\langle u^x\mid e_{i_0}\rangle\neq 0$. Then $(u^x,e_i,1\leq i\leq d,i\neq i_0)$ forms a new basis of $\mathbb{R}^d$. Applying the Gram-Schmidt orthonormalization procedure, we derive the existence of a new orthonormal basis $(u_1^x, ..., u_d^x)$ of $\mathbb{R}^d$ such that $u_1^x = u^x$. Moreover, the Gram-Schmidt orthonormalization procedure also implies that $u_i^x, 1 \le i \le d$ is continuous in x. With respect to this new basis $(u_1^x, ..., u_d^x)$, the hyperplane M_{12}^x defined in (35) can be written by

$$M_{12}^x = \frac{x_1 + x_2}{2} + \mathrm{span}\big(u_i^x,\ i = 2, ..., d\big),$$

where span(S) denotes the vector subspace of $\mathbb{R}^d$ spanned by S. Moreover, note that

$$V_1(x) \cap V_2(x) = \big\{\xi \in M_{12}^x \mid \min_{k=3,...,K} |x_k - \xi| \ge |x_1 - \xi| = |x_2 - \xi| \,\big\}.$$

Then, for every fixed $\xi \notin \partial\big(V_1(x) \cap V_2(x)\big)$, the function $x \mapsto \mathbb{1}_{V_1(x)\cap V_2(x)}(\xi)$ is continuous in $x \in F_K$ and

$$\lambda_x^{12}\Big(\partial\big(V_1(x) \cap V_2(x)\big)\Big) = 0 \tag{40}$$

since $V_1(x) \cap V_2(x)$ is a polyhedral convex set in M_{12}^x.

Now by a change of variable $\xi = \frac{x_1+x_2}{2} + \sum_{i=2}^d r_i u_i^x$,

$$h_{12}(x) = \int_{\mathbb{R}^{d-1}} \mathbb{1}_{V_{12}(x)}\big((r_2, ..., r_d)\big)\alpha\Big(x, \frac{x_1 + x_2}{2} + \sum_{i=2}^d r_i u_i^x\Big) dr_2...dr_d, \tag{41}$$

where

$$V_{12}(x) := \Big\{(r_2, ..., r_d) \in \mathbb{R}^{d-1} \;\Big|\; \min_{3\le k\le K} \Big|x_k - \frac{x_1 + x_2}{2} - \sum_{i=2}^d r_i u_i^x\Big| \ge \Big|\frac{x_1 - x_2}{2} - \sum_{i=2}^d r_i u_i^x\Big|\Big\}. \tag{42}$$

Let $\partial V_{12}(x)$ be the boundary of $V_{12}(x)$ given by

$$\partial V_{12}(x) := \Big\{(r_2, ..., r_d) \in \mathbb{R}^{d-1} \;\Big|\; \min_{3\le k\le K} \Big|x_k - \frac{x_1 + x_2}{2} - \sum_{i=2}^d r_i u_i^x\Big| = \Big|\frac{x_1 - x_2}{2} - \sum_{i=2}^d r_i u_i^x\Big|\Big\}.$$

Then (40) implies that $\lambda_{\mathbb{R}^{d-1}}\big(\partial V_{12}(x)\big) = 0$ where $\lambda_{\mathbb{R}^{d-1}}$ denotes the Lebesgue measure of the subspace span$\big(u_i^x, i = 2, ..., d\big)$.

It is obvious that for any $a = (a_1, ..., a_d), b = (b_1, ..., b_d) \in \mathbb{R}^d$, we have $|a_i b_j| \le |a|\,|b|\,, 1 \le i, j \le d$. Thus the absolute value of every term in the matrix

$$\begin{aligned}
&\alpha(x, \frac{x_1 + x_2}{2} + \sum_{i=2}^d r_i u_i^x) \\
&\quad = \frac{\big(\frac{x_1 - x_2}{2} - \sum_{i=2}^d r_i u_i^x\big) \otimes \big(\frac{x_2 - x_1}{2} - \sum_{i=2}^d r_i u_i^x\big)}{|x_2 - x_1|} f\Big(\frac{x_1 + x_2}{2} + \sum_{i=2}^d r_i u_i^x\Big)
\end{aligned}$$

can be upper-bounded by

$$\frac{\big|\frac{x_1 - x_2}{2} - \sum_{i=2}^d r_i u_i^x\big|\big|\frac{x_2 - x_1}{2} - \sum_{i=2}^d r_i u_i^x\big|}{|x_2 - x_1|} f\Big(\frac{x_1 + x_2}{2} + \sum_{i=2}^d r_i u_i^x\Big)$$

$$\leq \frac{\Big(\big|\frac{x_1-x_2}{2}\big| + \big|\sum_{i=2}^d r_i u_i^x\big|\Big)^2}{|x_2 - x_1|} f\Big(\frac{x_1+x_2}{2} + \sum_{i=2}^d r_i u_i^x\Big)$$

$$\leq C_x(1+\sum_{i=2}^d r_i^2) f\Big(\frac{x_1+x_2}{2} + \sum_{i=2}^d r_i u_i^x\Big) \tag{43}$$

where $C_x > 0$ is a constant depending only on x.

The distribution μ is assumed to be 1-radially controlled i.e. there exist a constant $A > 0$ and a continuous and decreasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\forall \xi \in \mathbb{R}^d, |\xi| \geq A, \qquad f(\xi) \leq g(|\xi|) \text{ and } \int_{\mathbb{R}_+} x^d g(x) dx < +\infty. \tag{44}$$

Now let $K := \frac{1}{2}|x_1 + x_2| \vee A$ and let $r := \sum_{i=2}^d r_i u_i^x$. As g is a non-increasing function, it follows that

$$C_x(1+\sum_{i=2}^d r_i^2) f\Big(\frac{x_1+x_2}{2} + \sum_{i=2}^d r_i u_i^x\Big)$$

$$\leq C_x(1+|r|^2) \sup_{\xi\in B(\mathbf{0},3K)} f(\xi)\mathbb{1}_{\{|r|\leq 2K\}} + C_x(1+|r|^2) g\Big(\Big|\frac{x_1^{(n)}+x_2^{(n)}}{2} + \sum_{i=2}^d r_i u_i^x\Big|\Big)\mathbb{1}_{\{|r|\geq 2K\}}.$$

$$\leq C_x(1+|r|^2) \sup_{\xi\in B(\mathbf{0},3K)} f(\xi)\mathbb{1}_{\{|r|\leq 2K\}} + C_x(1+|r|^2) g\big(|r| - K\big)\mathbb{1}_{\{|r|\geq 2K\}}.$$

Switching to polar coordinates, one obtains by letting $s = |r|$

$$\int_{\mathbb{R}^{d-1}} C_x |r|^2 g\big(|r| - K\big)\mathbb{1}_{\{|r|\geq 2K\}} dr_2...dr_d$$
$$\leq C_{x,d} \int_{\mathbb{R}_+} s^2 g(s-K)\mathbb{1}_{\{s\geq 2K\}} s^{d-2} ds \leq C_{x,d} \int_K^\infty (s+K)^d g(s) ds$$
$$\leq 2^d C_{x,d} \int_K^\infty (K^d + s^d) g(s) ds < +\infty,$$

where the last inequality follows from (44). Thus one obtains

$$\int_{\mathbb{R}^{d-1}} \Big[C_x(1+|r|^2) \sup_{\xi\in B(\mathbf{0},3K)} f(\xi)\mathbb{1}_{\{|r|\leq 2K\}} + C_x(1+|r|^2) g\big(|r| - K\big)\mathbb{1}_{\{|r|\geq 2K\}}\Big] dr_2...dr_d < +\infty.$$

Hence h_{12} is well-defined since

$$\int_{V_1(x)\cap V_2(x)} |\alpha(x,\xi)| \lambda_x^{12}(d\xi) < +\infty. \tag{45}$$

► *Step 2*: Now we prove that for any $x \in F_K$,

$$\sup_{y\in B(x,\varepsilon_x)} \Big|\frac{\partial \Phi_i^M}{\partial x_j}(y) - h_{ij}(y)\Big| \xrightarrow{M\to+\infty} 0, \tag{46}$$

where $\varepsilon_x = \frac{1}{3}\min_{1\le i<j\le K}|x_i - x_j|$ and (46) means every term in the matrix converges to 0.

First, for every fixed $y \in B(x, \varepsilon_x)$, the absolute value of every term in the following matrix

$$\frac{\partial \Phi_i^M}{\partial x_j}(y) - h_{ij}(y) = \int_{V_i(y)\cap V_j(y)} \frac{(y_i - \xi)\otimes(y_j - \xi)}{|y_j - y_i|} f(\xi)\big(1 - \chi_M(\xi)\big)\lambda_y^{ij}(d\xi)$$

can be upper bounded by

$$f_M(y) \coloneqq \int_{V_i(y)\cap V_j(y)\cap\left(\mathbb{R}^d\setminus B(0,M+1)\right)} \frac{|y_i - \xi||y_j - \xi|}{|y_j - y_i|} f(\xi)\lambda_y^{ij}(d\xi).$$

Moreover, the inequality (45) implies that $f_M(y)$ converges to 0 for every $y \in B(x, \varepsilon_x)$ as $M \to +\infty$. As $(f_M)_M$ is a monotonically decreasing sequence, one can obtain

$$\sup_{y\in B(x,\varepsilon)} \big|f_M(y)\big| \to 0$$

owing to Dini's theorem, which in turn implies the convergence in (46).

► *Step 3*: It is obvious that $\Phi_i^M(x)$ converges to $\Phi_i(x)$ for every $x \in \mathbb{R}^d$ as $M \to +\infty$ since $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. Hence $\frac{\partial \Phi_1}{\partial x_2}(x) = h_{12}(x)$. Then one can directly obtain (16) since $\frac{\partial \mathcal{D}_{K,\mu}}{\partial x_j x_i} = 2\frac{\partial \Phi_i}{\partial x_j} = 2h_{ij}$ by applying (15). The proof for (17) is similar.

(ii) We will only prove the continuity of $\frac{\partial^2 \mathcal{D}_{K,\mu}}{\partial x_1 \partial x_2}$ and $\frac{\partial^2 \mathcal{D}_{K,\mu}}{\partial x_1^2}$ at a point $x \in F_K$. The proof for $\frac{\partial^2 \mathcal{D}_{K,\mu}}{\partial x_i \partial x_j}$ for others $i, j \in \{1, ..., K\}$ is similar. We take the same definition of $\alpha(x, \xi)$ in (39), then

$$\frac{\partial^2 \mathcal{D}_{K,\mu}}{\partial x_1 \partial x_2}(x) = 2\int_{V_1(x)\cap V_2(x)} \alpha(x,\xi)\lambda_x^{12}(d\xi)$$

and by the same change of variable (41) as in (i), we have

$$\frac{\partial^2 \mathcal{D}_{K,\mu}}{\partial x_1 \partial x_2}(x) = 2\int_{\mathbb{R}^{d-1}} \mathbb{1}_{V_{12}(x)}\big((r_2, ..., r_d)\big)\alpha\Big(x, \frac{x_1 + x_2}{2} + \sum_{i=2}^{d} r_i u_i^x\Big) dr_2...dr_d$$

with the same definition of $V_{12}(x)$ as in (42).

Let us now consider a sequence $x^{(n)} = (x_1^{(n)}, ..., x_K^{(n)}) \in (\mathbb{R}^d)^K$ converging to a point $x = (x_1, ..., x_K) \in F_K$ satisfying that for every $n \in \mathbb{N}^*$,

$$\big|x^{(n)} - x\big| \le \delta_x \coloneqq \frac{1}{3}\min_{1\le i,j\le K, i\ne j} |x_i - x_j|\,, \tag{47}$$

so that $x^{(n)} \in F_K$ for every $n \in \mathbb{N}^*$. For a fixed $(r_2, ..., r_d) \in \mathbb{R}^{d-1}$, the continuity of $x \mapsto \alpha(x, \frac{x_1+x_2}{2} + \sum_{i=2}^d r_i u_i^x)$ in F_K can be obtained by the continuity of $(x, \xi) \mapsto \alpha(x, \xi)$ and the continuity of Gram-Schmidt orthonormalization procedure.

By the same reasoning as in (43), the absolute value of every term in the matrix

$$\alpha\Big(x^{(n)}, \frac{x_1^{(n)} + x_2^{(n)}}{2} + \sum_{i=2}^{d} r_i^{(n)} u_i^{x^{(n)}}\Big)$$

can be upper bounded by

$$\frac{\Big(\big|\frac{x_1^{(n)}-x_2^{(n)}}{2}\big|+\big|\sum_{i=2}^d r_i u_i^{x^{(n)}}\big|\Big)^2}{\Big|x_2^{(n)}-x_1^{(n)}\Big|} f\Big(\frac{x_1^{(n)}+x_2^{(n)}}{2}+\sum_{i=2}^d r_i^{(n)} u_i^{x^{(n)}}\Big),$$

where there exists a constant C_x depending only on x such that

$$\frac{\Big(\big|\frac{x_1^{(n)}-x_2^{(n)}}{2}\big|+\big|\sum_{i=2}^d r_i u_i^{x^{(n)}}\big|\Big)^2}{\Big|x_2^{(n)}-x_1^{(n)}\Big|} \leq C_x(1+\sum_{i=2}^d r_i^2)$$

since by (47), one can get

$$\forall n\in\mathbb{N}^*, \forall i,j\in\{1,...,K\} \text{ with } i\neq j, \quad \delta_x \leq \Big|x_i^{(n)}-x_j^{(n)}\Big| \leq \max_{1\leq i,j\leq K}|x_i-x_j|+2\delta_x.$$

Moreover, if we take $K \coloneqq \frac{1}{2}\sup_n\Big|x_1^{(n)}+x_2^{(n)}\Big|\vee A$ and take $r_n \coloneqq \sum_{i=2}^d r_i u_i^{x^{(n)}}$, then

$$\begin{aligned}
&C_x(1+\sum_{i=2}^d r_i^2) f\Big(\frac{x_1^{(n)}+x_2^{(n)}}{2}+\sum_{i=2}^d r_i u_i^{x^{(n)}}\Big)\\
&\leq C_x(1+|r|^2)\sup_{\xi\in B(\mathbf{0},3K)} f(\xi)\mathbb{1}_{\{|r|\leq 2K\}} + C_x(1+|r|^2) g\Big(\Big|\frac{x_1^{(n)}+x_2^{(n)}}{2}+\sum_{i=2}^d r_i u_i^{x^{(n)}}\Big|\Big)\mathbb{1}_{\{|r|\geq 2K\}}.\\
&\leq C_x(1+|r|^2)\sup_{\xi\in B(\mathbf{0},3K)} f(\xi)\mathbb{1}_{\{|r|\leq 2K\}} + C_x(1+|r|^2) g\big(|r|-K\big)\mathbb{1}_{\{|r|\geq 2K\}}.
\end{aligned}$$

By the same reasoning as in (i)-Step 1, we have

$$\int_{\mathbb{R}^{d-1}}\Big[C_x(1+|r|^2)\sup_{\xi\in B(\mathbf{0},3K)} f(\xi)\mathbb{1}_{\{|r|\leq 2K\}} + C_x(1+|r|^2) g\big(|r|-K\big)\mathbb{1}_{\{|r|\geq 2K\}}\Big] dr_2...dr_d < +\infty,$$

which implies $\frac{\partial^2\mathcal{D}_{K,\mu}}{\partial x_1\partial x_2}(x^{(n)}) \to \frac{\partial^2\mathcal{D}_{K,\mu}}{\partial x_1\partial x_2}(x)$ as $n\to+\infty$ by applying Lebesgue's dominated convergence theorem. Thus $\frac{\partial^2\mathcal{D}_{K,\mu}}{\partial x_1\partial x_2}$ is continuous at $x\in F_K$.

It remains to prove the continuity of $x\mapsto\mu\big(V_1(x)\big)=\int_{\mathbb{R}^d}\mathbb{1}_{V_1(x)}(\xi)f(\xi)\lambda_d(d\xi)$ to obtain the continuity of $\frac{\partial^2\mathcal{D}_{K,\mu}}{\partial x_1^2}$ defined in (17). Remark that

$$V_1(x)=\Big\{\xi\in\mathbb{R}^d \;\Big|\; |\xi-x_1|\leq\min_{1\leq j\leq K}|\xi-x_j|\Big\},$$

and by Graf and Luschgy (2000, Proposition 1.3),

$$\partial V_1(x)=\Big\{\xi\in\mathbb{R}^d \;\Big|\; |\xi-x_1|=\min_{1\leq j\leq K}|\xi-x_j|\Big\}.$$

Then for any $\xi\notin\partial V_1(x)$, the function $x\mapsto\mathbb{1}_{V_1(x)}(\xi)$ is continuous. As the norm $|\cdot|$ is the Euclidean norm, then $\lambda_d(\partial V_i(x))=0$ (see Graf and Luschgy, 2000, Proposition 1.3 and Theorem 1.5). For any $x\in F_K$ and a sequence $x^{(n)}$ converging to x, we have $\mathbb{1}_{V_1(x^{(n)})}(\xi)f(\xi)\leq f(\xi)\in L^1(\lambda_d)$. Thus the continuity of $x\mapsto\mu\big(V_1(x)\big)=\int_{\mathbb{R}^d}\mathbb{1}_{V_1(x)}(\xi)f(\xi)\lambda_d(d\xi)$ is a direct application of Lebesgue's dominated convergence theorem. ■

Appendix D: Proof of Proposition 11

Proof [Proof of Proposition 11] (i) We will only deal with the uniform distribution $U([0,1])$. The proof is similar for other uniform distributions.

In Graf and Luschgy (2000, Example 4.17 and 5.5) and Benaïm et al. (1998), the authors show that $\Gamma^* = \{\frac{2i-1}{2K} : i-1, ..., K\}$ is the unique optimal quantizers of $U([0,1])$. Let $x^* = (\frac{1}{2K}, ..., \frac{2i-1}{2K}, ..., \frac{2K-1}{2K})$, then one can compute explicitly $H_{\mathcal{D}}(x^*)$:

$$H_{\mathcal{D}}(x^*) = \begin{bmatrix} \frac{3}{2K} & -\frac{1}{2K} & & & & & 0 \\ & \ddots & \ddots & \ddots & & & \\ & & -\frac{1}{2K} & \frac{1}{K} & -\frac{1}{2K} & & \\ & & & \ddots & \ddots & \ddots & \\ 0 & & & & & -\frac{1}{2K} & \frac{3}{2K} \end{bmatrix},$$

The matrix $H_{\mathcal{D}}(x^*)$ is tridiagonal. If we denote by $f_k(x^*)$ its k-th leading principal minor and we define $f_0(x^*) = 1$, then

$$f_k(x^*) = \frac{1}{K} f_{k-1}(x^*) - \frac{1}{4K^2} f_{k-2}(x^*) \quad \text{for } k = 2, ..., K-1, \tag{48}$$

and $f_1(x^*) = \frac{3}{2K}$ and $f_K(x^*) = |H_{\mathcal{D}}(x^*)| = \frac{3}{K} f_{K-1}(x^*) - \frac{1}{4K^2} f_{K-2}(x^*)$ (see El-Mikkawy, 2003). One can solve from the three-term recurrence relation that

$$\begin{aligned} f_k(x^*) &= \frac{2k+1}{2^k K^k}, \quad \text{for } k = 1, ..., K-1 \\ \text{and} \quad f_K(x^*) &= \frac{2K+1}{2^K K^K} + \frac{1}{2K} f_{K-1}. \end{aligned} \tag{49}$$

In fact, (49) is true for $k = 1$. Suppose (49) holds for $k \leq K-2$, then owing to (48)

$$f_{k+1}(x^*) = \frac{1}{K} \cdot \frac{2k+1}{2^k K^k} - \frac{1}{4K^2} \cdot \frac{2(k-1)+1}{2^{k-1} K^{k-1}} = \frac{2(k+1)+1}{2^{k+1} K^{k+1}}.$$

Then it is obvious that $f_k(x^*) > 0$ for $k = 1, ..., K$. Thus, $H_{\mathcal{D}}(x^*)$ is positive definite.

(ii) We define for $i = 2, ..., K$, $\widetilde{x}_i^* = \frac{x_{i-1}^* + x_i^*}{2}$, then the Voronoi region $V_i(x^*) = [\widetilde{x}_i^*, \widetilde{x}_{i+1}^*]$ for $i = 2, ..., K-1$, $V_1(x^*) = (-\infty, \widetilde{x}_2^*]$ and $V_K(x^*) = [\widetilde{x}_K^*, +\infty)$.

For $2 \leq i \leq K-1$,

$$\begin{aligned} L_i(x^*) &= A_i - 2B_{i-1,i} - 2B_{i,i+1} \\ &= 2\mu\big(V_i(x^*)\big) - (x_i^* - x_{i-1}^*) f\Big(\frac{x_{i-1}^* + x_i^*}{2}\Big) - (x_{i+1}^* - x_i^*) f\Big(\frac{x_i^* + x_{i+1}^*}{2}\Big) \\ &= 2\mu\big(V_i(x^*)\big) - 2(x_i^* - \widetilde{x}_i^*) f(\widetilde{x}_i^*) - 2(\widetilde{x}_{i+1}^* - x_i^*) f(\widetilde{x}_{i+1}^*) \\ &= \frac{2}{\mu\big(V_i(x^*)\big)} \Big\{ \mu\big(V_i(x^*)\big)^2 - [x_i^* \mu\big(V_i(x^*)\big) \\ &\qquad - \widetilde{x}_i^* \mu\big(V_i(x^*)\big)] f(\widetilde{x}_i^*) - [\widetilde{x}_{i+1}^* \mu\big(V_i(x^*)\big) - x_i^* \mu\big(V_i(x^*)\big)] f(\widetilde{x}_{i+1}^*) \Big\} \\ &= \frac{2}{\mu\big(V_i(x^*)\big)} \Big\{ \mu\big(V_i(x^*)\big)^2 - [\int_{V_i(x^*)} \xi f(\xi) d\xi - \widetilde{x}_i^* \int_{V_i(x^*)} f(\xi) d\xi] f(\widetilde{x}_i^*) \end{aligned}$$

$$- [\widetilde{x}^*_{i+1} \int_{V_i(x^*)} f(\xi)d\xi - \int_{V_i(x^*)} \xi f(\xi)d\xi] f(\widetilde{x}^*_{i+1})\Big\} \quad \big(\text{owing to (18)}\big)$$
$$= \frac{2}{\mu\big(V_i(x^*)\big)}\Big\{\mu\big(V_i(x^*)\big)^2 - f(\widetilde{x}^*_i)\int_{V_i(x^*)}(\xi-\widetilde{x}^*_i)f(\xi)d\xi + f(\widetilde{x}^*_{i+1})\int_{V_i(x^*)}(\xi-\widetilde{x}^*_{i+1})f(\xi)d\xi\Big\}.$$

In order to study the positivity of $L_i(x^*)$, we define a function $\varphi_i(u)$ for any $i \in \{1, ..., K\}$ and for any $u = (u_1, ..., u_{K+1}) \in F^+_{K+1}$ by

$$\varphi_i(u) := \big[\int_{u_i}^{u_{i+1}} f(\xi)d\xi\big]^2 - f(u_i)\int_{u_i}^{u_{i+1}}(\xi-u_i)f(\xi)d\xi + f(u_{i+1})\int_{u_i}^{u_{i+1}}(\xi-u_{i+1})f(\xi)d\xi, \quad (50)$$

Lemma 17 *If f is positive and differentiable and if $\log f$ is strictly concave, then for all $u = (u_1, ..., u_{K+1}) \in F^+_{K+1}$, we have the following results for $\varphi_i(u)$ defined in (50),*

(a) for every $i = 1, ..., K$, $\varphi_i(u) > 0$;

(b) $\frac{\partial \varphi_1}{\partial u_1}(u) < 0$;

(c) $\frac{\partial \varphi_K}{\partial u_{K+1}}(u) > 0$.

Proof [Proof of lemma 17] For a fixed $i \in \{1, ..., K\}$, the partial derivatives of φ_i are

$$\frac{\partial \varphi_i}{\partial u_i}(u) = -2\big[\int_{u_i}^{u_{i+1}} f(\xi)d\xi\big] f(u_i) - f'(u_i)\int_{u_i}^{u_{i+1}}(\xi-u_i)f(\xi)d\xi + f(u_i)f(u_{i+1})(u_{i+1}-u_i)$$
$$\frac{\partial \varphi_i}{\partial u_{i+1}}(u) = 2\big[\int_{u_i}^{u_{i+1}} f(\xi)d\xi\big] f(u_{i+1}) + f'(u_{i+1})\int_{u_i}^{u_{i+1}}(\xi-u_{i+1})f(\xi)d\xi$$
$$- f(u_i)f(u_{i+1})(u_{i+1}-u_i)$$
$$\frac{\partial \varphi_i}{\partial u_l}(u) = 0, \quad \text{for all } l \neq i \text{ and } l \neq i+1.$$

The second derivatives of φ_i are

$$\frac{\partial^2 \varphi_i}{\partial u_{i+1}\partial u_i}(u) = \frac{\partial^2 \varphi_i}{\partial u_i \partial u_{i+1}}(u) = -f(u_{i+1})f(u_i) + (u_{i+1}-u_i)\big(f(u_i)f'(u_{i+1}) - f'(u_i)f(u_{i+1})\big)$$
$$\frac{\partial^2 \varphi_i}{\partial u_l \partial u_i}(u) = \frac{\partial^2 \varphi_i}{\partial u_i \partial u_l}(u) = 0 \quad \text{for all } l \neq i \text{ and } l \neq i+1.$$

If $\log f$ is strictly concave, then $(\log f)' = \dfrac{f'}{f}$ is strictly decreasing. For $u \in F^+_{K+1}$, we have $u_{i+1} > u_i$, then

$$\frac{f'(u_{i+1})}{f(u_{i+1})} - \frac{f'(u_i)}{f(u_i)} = \frac{f'(u_{i+1})f(u_i) - f(u_{i+1})f'(u_i)}{f(u_i)f(u_{i+1})} < 0.$$

Thus $f'(u_{i+1})f(u_i) - f(u_{i+1})f'(u_i) < 0$ and from which one can get $\dfrac{\partial^2 \varphi_i}{\partial u_{i+1}\partial u_i}(u) < 0$.

In fact, φ_i, $\frac{\partial \varphi_i}{\partial u_i}$, $\frac{\partial \varphi_i}{\partial u_{i+1}}$ and $\frac{\partial^2 \varphi_i}{\partial u_{i+1}\partial u_i}$ only depend on the variables u_i and u_{i+1}.

(a) For $1 \leq i \leq K$, $\varphi_i(u_{i+1}, u_{i+1}) = 0$. After the Mean value theorem, there exists $\gamma \in (u_i, u_{i+1})$ such that

$$\frac{1}{u_i - u_{i+1}}\big(\varphi_i(u_i, u_{i+1}) - \varphi_i(u_{i+1}, u_{i+1})\big) = \frac{\partial \varphi_i}{\partial u_i}(\gamma, u_{i+1}). \tag{51}$$

Moreover, there exists $\zeta \in (\gamma, u_{i+1})$ such that

$$\frac{1}{u_{i+1} - \gamma}\big(\frac{\partial \varphi_i}{\partial u_i}(\gamma, u_{i+1}) - \frac{\partial \varphi_i}{\partial u_i}(\gamma, \gamma)\big) = \frac{\partial^2 \varphi_i}{\partial u_{i+1} \partial u_i}(\gamma, \zeta).$$

As $\gamma < \zeta$, $\frac{\partial^2 \varphi_i}{\partial u_{i+1} \partial u_i}(\gamma, \zeta) < 0$. Thus $\frac{\partial \varphi_i}{\partial u_i}(\gamma, u_{i+1}) < 0$, since $\frac{\partial \varphi_i}{\partial u_i}(\gamma, \gamma) = 0$. Then $\varphi_i(u_i, u_{i+1}) > 0$ by applying $\frac{\partial \varphi_i}{\partial u_i}(\gamma, u_{i+1}) < 0$ in (51).

(b) After the Mean value theorem, there exists $\gamma' \in (u_1, u_2)$ such that

$$\frac{\partial^2 \varphi_1}{\partial u_1 \partial u_2}(u_1, \gamma') = \frac{1}{u_2 - u_1}\big(\frac{\partial \varphi_1}{\partial u_1}(u_1, u_2) - \frac{\partial \varphi_1}{\partial u_1}(u_1, u_1)\big).$$

As $\frac{\partial^2 \varphi_1}{\partial u_1 \partial u_2}(u_1, \gamma') < 0$ and $\frac{\partial \varphi_1}{\partial u_1}(u_1, u_1) = 0$, one can get $\frac{\partial \varphi_1}{\partial u_1}(u_1, u_2) < 0$.

(c) In the same way, there exists $\zeta' \in (u_K, u_{K+1})$ such that

$$\frac{\partial^2 \varphi_K}{\partial u_K \partial u_{K+1}}(\zeta', u_{K+1}) = \frac{1}{u_K - u_{K+1}}\big(\frac{\partial \varphi_K}{\partial u_{K+1}}(u_K, u_{K+1}) - \frac{\partial \varphi_K}{\partial u_{K+1}}(u_{K+1}, u_{K+1})\big).$$

As $\frac{\partial^2 \varphi_K}{\partial u_K \partial u_{K+1}}(\zeta', u_{K+1}) < 0$ and $\frac{\partial \varphi_K}{\partial u_{K+1}}(u_{K+1}, u_{K+1}) = 0$, one gets $\frac{\partial \varphi_K}{\partial u_{K+1}}(u_K, u_{K+1}) > 0$. ■

Proof [Proof of Proposition 11, continuation]

We set $\widetilde{x}^{*,M} := (-M, \widetilde{x}_2^*, ..., \widetilde{x}_K^*, M)$ with M large enough such that $\widetilde{x}^{*,M} \in F_{K+1}^+$, then for $2 \leq i \leq K-1$, $L_i(x^*) = \frac{2}{\mu(V_i(x^*))}\varphi_i(\widetilde{x}^{*,M})$. Thus $L_i(x^*) > 0,\ i = 2, ..., K-1$ owing to Lemma 17-(a).

For $i = 1$,

$$\begin{aligned} L_1(x^*) &= A_1(x^*) - 2B_{1,2}(x^*) \\ &= \frac{2}{\mu\big(V_1(x^*)\big)}\Big\{\mu\big(V_1(x^*)\big)^2 - f(\widetilde{x}_2^*)\int_{V_1(x^*)}(\widetilde{x}_2^* - \xi)f(\xi)d\xi\Big\}. \end{aligned}$$

If we denote $D_1(x^*) := \mu\big(V_1(x^*)\big)^2 - f(\widetilde{x}_2^*)\int_{V_1(x^*)}(\widetilde{x}_2^* - \xi)f(\xi)d\xi$, then

$$D_1(x^*) = \lim_{M \to +\infty} \varphi_1(\widetilde{x}^{*,M}) + f(-M)\int_{V_1^M(x^*)}\big(\xi - (-M)\big)f(\xi)d\xi,$$

where $V_1^M(x^*) = [-M, \widetilde{x}_2^*]$.

For all M such that $-M < \widetilde{x}_2^*$, $f(-M)\int_{V_1^M(x^*)}\big(\xi-(-M)\big)f(\xi)d\xi > 0$, then

$$\lim_{M\to+\infty} f(-M)\int_{V_1^M(x^*)}\big(\xi-(-M)\big)f(\xi)d\xi \geq 0.$$

It follows from Lemma 17-(b) that $\frac{\partial \varphi_1}{\partial u_1}(u) < 0$ for $u \in F_{K+1}^+$, so that for a fixed M_1 such that $\widetilde{x}^{*,M_1} \in F_{K+1}^+$, we have $\varphi_1(\widetilde{x}^{*,M_1}) \leq \lim_{M\to+\infty}\varphi_1(\widetilde{x}^{*,M})$. We also have $\varphi_1(\widetilde{x}^{*,M_1}) > 0$ by applying Lemma 17-(a). It follows that

$$\begin{aligned} D_1(x^*) &= \lim_{M\to+\infty}\varphi_1(\widetilde{x}^{*,M}) + \lim_{M\to+\infty} f(-M)\int_{V_1^M(x^*)}\big(\xi-(-M)\big)f(\xi)d\xi \\ &\geq \varphi_1(\widetilde{x}^{*,M_1}) + \lim_{M\to+\infty} f(-M)\int_{V_1^M(x^*)}\big(\xi-(-M)\big)f(\xi)d\xi \\ &> 0. \end{aligned}$$

Then $L_1(x^*) = \frac{2}{\mu\big(V_1(x^*)\big)}D_1(x^*) > 0$.

The proof of $L_K(x^*)$ is similar by applying Lemma 17-(c). Thus $H_{\mathcal{D}}(x^*)$ is positive definite owing to Gershgorin circle theorem. ■

Appendix E: Proof of Theorem 15 - (b) and (c)

Proof (b) If μ has a c-th polynomial tail with $c > d+p$, then $\mu \in \mathcal{P}_p(\mathbb{R}^d)$. Let $X, X_1, ..., X_n$ be i.i.d random variable with probability distribution μ. Then,

$$\begin{aligned} r_n = \|R_n\|_2^2 = \mathbb{E}\big[\max\left(|X_1|, ..., |X_n|\right)^2\big] &= \mathbb{E}\big[\max(|X_1|^p, ..., |X_n|^p)^{2/p}\big] \\ \leq \mathbb{E}\Big(\big[\sum_{i=1}^n |X_i|^p\big]^{2/p}\Big) \leq \Big[\mathbb{E}\big(\sum_{i=1}^n |X_i|^p\big)\Big]^{2/p} &= \Big[n\,\mathbb{E}\,|X|^p\Big]^{2/p} = n^{2/p}\,\|X\|_p^2, \end{aligned}$$

where the last line is due to the fact that $X_1, ..., X_n$ have the same distribution as X. Moreover, we have

$$\rho_K(\mu) = K^{\frac{p+d}{d(c-p-d)}\gamma_K} \quad \text{with} \quad \lim_{K\to+\infty}\gamma_K = 1 \tag{52}$$

owing to (11). It follows from (30) that

$$\mathbb{E}\big[\mathcal{D}(x^{(n)}) - \mathcal{D}(x)\big] \leq \frac{2K}{\sqrt{n}}\Big[3r_{2n}^2 + \big((2m_2)\vee\rho_K(\mu)\big)\cdot\rho_K(\mu)\Big]$$

since $r_{2n} \geq m_2$ after the definitions of r_{2n} and m_2. In addition, (52) implies that $\rho_K(\mu) \to +\infty$ as $K \to +\infty$ and, for large enough K, $\rho_K(\mu) \geq 2m_2$. Therefore,

$$\mathbb{E}\big[\mathcal{D}(x^{(n)}) - \mathcal{D}(x)\big] \leq \frac{2K}{\sqrt{n}}\Big(3\cdot(2n)^{2/p}\,\|X\|_p^2 + 3K^{\frac{p+d}{d(c-p-d)}\gamma_K}\Big)$$

$$= \frac{K}{\sqrt{n}}\Big(C_{\mu,p}\, n^{2/p} + 6K^{\frac{p+d}{d(c-p-d)}}\gamma_K\Big),$$

where $C_{\mu,p} = 6 \cdot 2^{2/p} \left\| X \right\|_p^2$ and $\lim_K \gamma_K = 1$.
(c) The distribution μ is assumed to have a hyper-exponential tail, that is, $\mu = f \cdot \lambda_d$ with $f(\xi) = \tau \left|\xi\right|^c e^{-\vartheta|\xi|^\kappa}$ for $|\xi|$ large enough with $c > -d$. The real constant κ is assumed to be greater than or equal to 2. Let X be a random variable with probability distribution μ. Therefore, for every $\lambda \in (0, \vartheta)$, $\mathbb{E}\, e^{\lambda|X|^\kappa} < +\infty$ and

$$\begin{aligned}
r_n = \|R_n\|_2^2 &= \mathbb{E}\big[\max(|X_1|, ..., |X_n|)^2\big] = \mathbb{E}\big[\max(|X_1|^\kappa, ..., |X_n|^\kappa)^{2/\kappa}\big] \\
&= \mathbb{E}\Big(\Big[\frac{1}{\lambda}\log\big(\max(e^{\lambda|X_1|^\kappa}, ..., e^{\lambda|X_n|^\kappa})\big)\Big]^{2/\kappa}\Big) \leq \Big(\frac{1}{\lambda}\Big)^{2/\kappa}\Big[\log \mathbb{E}\max(e^{\lambda|X_1|^\kappa}, ..., e^{\lambda|X_n|^\kappa})\Big]^{2/\kappa} \\
&\leq \Big(\frac{1}{\lambda}\Big)^{2/\kappa}\Big\{\log \mathbb{E}\Big[\sum_{i=1}^n e^{\lambda|X_i|^\kappa}\Big]\Big\}^{2/\kappa} = \Big(\frac{1}{\lambda}\Big)^{2/\kappa}\big\{\log(n\mathbb{E}\, e^{\lambda|X|^\kappa})\big\}^{2/\kappa} \\
&= \Big(\frac{1}{\lambda}\Big)^{2/\kappa}\big(\log \mathbb{E}\, e^{\lambda|X|^\kappa} + \log n\big)^{2/\kappa},
\end{aligned} \tag{53}$$

where the last line of (53) is due to the fact that $X_1, ..., X_n$ have the same distribution than X. Under the same assumption as before, it follows from (12) that

$$\rho_K(\mu) \leq \gamma_K (\log K)^{1/\kappa} \cdot 2\vartheta^{-1/\kappa}\big(1 + \frac{2}{d}\big)^{1/\kappa} \ \text{ with } \limsup_{K\to+\infty} \gamma_K \leq 1. \tag{54}$$

Moreover, it follows from (30) that

$$\mathbb{E}\big[\mathcal{D}(x^{(n)}) - \mathcal{D}(x)\big] \leq \frac{2K}{\sqrt{n}}\Big[3r_{2n}^2 + \big((2m_2) \vee \rho_K(\mu)\big) \cdot \rho_K(\mu)\Big]$$

since $r_{2n} \geq m_2$ after the definitions of r_{2n} and m_2. In addition, (54) implies that $\rho_K(\mu) \to +\infty$ as $K \to +\infty$ and, for large enough K, $\rho_K(\mu) \geq 2m_2$. Therefore,

$$\begin{aligned}
\mathbb{E}\big[\mathcal{D}(x^{(n)}) - \mathcal{D}(x)\big] \leq & \frac{2K}{\sqrt{n}}\Big\{3 \cdot \Big(1 \vee \log\big(2\mathbb{E}\, e^{\lambda|X|^\kappa}\big)\Big)^{2/\kappa}\big(\frac{1}{\lambda}\big)^{2/\kappa}\big[(\log n)^{2/\kappa} + 1\big]\Big\} \\
& + 4\vartheta^{-2/\kappa}\gamma_K(\log K)^{2/\kappa}\big(1 + \frac{2}{d}\big)^{2/\kappa}.
\end{aligned} \tag{55}$$

Inequality (55) is true for all $\lambda \in (0, \vartheta)$. We may take $\lambda = \frac{\vartheta}{2}$. It follows that

$$\mathbb{E}\big[\mathcal{D}(x^{(n)}) - \mathcal{D}(x)\big] \leq C_{\vartheta,\kappa,\mu} \cdot \frac{K}{\sqrt{n}}\Big[1 + (\log n)^{2/\kappa} + \gamma_K(\log K)^{2/\kappa}\big(1 + \frac{2}{d}\big)^{2/\kappa}\Big],$$

where $C_{\vartheta,\kappa,\mu} = \Big[6\big(\frac{2}{\vartheta}\big)^{2/\kappa} \cdot (1 \vee \log 2\mathbb{E}\, e^{\vartheta|X|^\kappa/2})\Big] \vee 8\vartheta^{-2/\kappa}$ and $\limsup_K \gamma_K = 1$.

Multi-dimensional normal distribution is a special case of hyper-exponential tail distribution, i.e. if $\mu = \mathcal{N}(m, \Sigma)$, we have $\kappa = 2, \vartheta = \frac{1}{2}$ and $c = 0$. By the same reasoning as before,

$$\mathbb{E}\big[\mathcal{D}(x^{(n)}) - \mathcal{D}(x)\big] \leq C_\mu \cdot \frac{K}{\sqrt{n}}\Big[1 + \log n + \gamma_K \log K\big(1 + \frac{2}{d}\big)\Big],$$

where $C_\mu = 24 \cdot \big(1 \vee \log 2\mathbb{E}\, e^{|X|^2/4}\big)$. When $\mu = \mathcal{N}(0, \mathbf{I}_d)$, $C_\mu = 24(1 + \frac{d}{2}) \cdot \log 2$, since $\mathbb{E}\, e^{|X|^2/4} = 2^{d/2}$ by the moment-generating function of a χ^2 distribution. ■

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