

Nonparametric Inference under B-bits Quantization

Kexuan Li

*Global Biometrics and Data Sciences
Bristol Myers Squibb
Princeton Pike, NJ 08648, USA*

KEXUAN.LI.77@GMAIL.COM

Ruiqi Liu

*Department of Mathematics and Statistics
Texas Tech University
Lubbock, TX 79409, USA*

RUIQLIU@TTU.EDU

Ganggang Xu

*Department of Management Science
University of Miami
Coral Gables, FL 33146, USA*

GANGXU@BUS.MIAMI.EDU

Zuofeng Shang

*Department of Mathematical Sciences
New Jersey Institute of Technology
Newark, NJ 07102, USA*

ZSHANG@NJIT.EDU

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Abstract

Statistical inference based on lossy or incomplete samples is often needed in research areas such as signal/image processing, medical image storage, remote sensing, signal transmission. In this paper, we propose a nonparametric testing procedure based on samples quantized to B bits through a computationally efficient algorithm. Under mild technical conditions, we establish the asymptotic properties of the proposed test statistic and investigate how the testing power changes as B increases. In particular, we show that if B exceeds a certain threshold, the proposed nonparametric testing procedure achieves the classical minimax rate of testing (Shang and Cheng, 2015) for spline models. We further extend our theoretical investigations to a nonparametric linearity test and an adaptive nonparametric test, expanding the applicability of the proposed methods. Extensive simulation studies together with a real-data analysis are used to demonstrate the validity and effectiveness of the proposed tests.

Keywords: B-bits Quantization, Minimax Rates of Testing, Nonparametric Inference, Smoothing Splines

1. Introduction

Lossy or incomplete data are commonly encountered in research areas such as machine learning, information theory, and signal processing. To store and process signals in digital devices, *quantization* is a popular procedure that maps the original measurements from a large (often uncountably infinite) set to a set of possible values. The resulting values are referred to as the *quantized samples*. With the increasing availability of data, it is of great

interest to quantify how the data analysis can be affected when the data are quantized due to storage or communication budget constraint, and how to design quantization schemes to minimize the efficiency loss. Statistical inference based on quantized samples is challenging because, in addition to the measurement errors, one also needs to account for the information loss due to the quantization errors. In particular, commonly used standard statistical procedures may not be valid when applied to quantized samples if the quantization errors are ignored.

The research on lossy data has attracted increasing attention recently. The first line of works focuses on b -bit compressive sensing, which aims at reconstructing a sparse signal from a sequence of b -bit quantized outcomes. A 1-bit compressive sensing model was proposed by Boufounos and Baraniuk (2008), and several efficient and provable algorithms have been developed; see, e.g., Gupta et al. (2010); Gopi et al. (2013); Plan and Vershynin (2013); Zhang et al. (2014); Zhu and Gu (2015). A signal recovery algorithm was proposed in Slawski and Li (2015), which extended the 1-bit compressive sensing model to a b -bit compressive sensing model. The second line of research related to the lossy data is to develop statistical methods based on quantized observations. For example, Lee and Vardeman (2001) studied the interval estimation of a normal mean process from rounded data, which was further extended to more general likelihood-based statistical estimation problems (Vardeman and Lee, 2005) and nonparametric regression problems (Benhenni and Rachdi, 2006). Recently, an increasing number of works aim to quantify the impact of quantization on the statistical properties of the resulting estimators. For example, Zhang et al. (2013) established lower bounds on the minimax risks for distributed estimation of parametric models under a communication budget constraint. Suresh et al. (2017) proposed communication efficient algorithms for distributed mean estimation without probabilistic assumptions on the data. A version of Pinsker's theorem under some storage or communication constraints was developed in Zhu and Lafferty (2014), and it was further applied to analyze the convergence rate of nonparametric estimation with a limited bits budget by Zhu and Lafferty (2017). More recently, a series of works have emerged in investigating the high-dimensional and/or nonparametric regression model estimation in the distributed learning framework with bits constraints, e.g., see Zhu and Lafferty (2018); Han et al. (2018); Szabo and van Zanten (2020); Cai and Wei (2021).

Despite the abundant existing literature on statistical modeling of quantized data, research focusing on the nonparametric inference based on quantized data is still lacking. This paper aims to fill this gap by proposing a new quantization scheme with a B -bits storage or communication budget such that nonparametric estimation and testing based on quantized samples are still valid. Specifically, we consider the following regression model

$$y_i = g_0(i/n) + \sigma\epsilon_i, \quad i = 1, \dots, n, \quad (1)$$

where $g_0(\cdot)$ is a smooth function, ϵ_i 's are *iid* zero-mean errors with an unit variance, and $\sigma > 0$ is an unknown constant. The goal is to (a) estimate $g_0(\cdot)$, and (b) test the following hypothesis

$$H_0 : g_0(x) = g_*(x) \text{ for all } x \in [0, 1], \quad (2)$$

where $g_*(\cdot)$ is a pre-specified deterministic function.

The above model has been extensively studied in the literature, see, e.g., Shang and Cheng (2017), and is closely related to the well-known Gaussian sequence model and Gaussian white noise model (Tsybakov, 2008). However, unlike existing literature, we consider the case in which the original data, denoted by y_1, \dots, y_n , are generated in machine M, and are quantized as soon as they are generated. The quantized data are then stored in a machine M or transmitted to another machine M* for future statistical inferences. We assume that only B -bits budget are available for data storage or communication, rendering the necessity for data quantization that may invalidate existing estimation and inference methods. Such a research problem is important for applications where data generation and analysis are carried out at different locations. For example, testing $H_0 : g_0(x) = 0$ reveals whether the transmitted quantized signals through satellite are pure noises. If $g_*(\cdot)$ is the signal-process from a normally functioning machine, testing (2) using only quantized samples enables us to remotely monitor whether the machine is working properly in real-time.

To meet the B -bits requirement, we propose a two-stage quantization procedure: in the first stage we quantize an individual y_i as $Q(y_i)$ with $Q(\cdot)$ being a quantizer, and in the second stage we overwrite these quantized observations by their local averages. See Figure 1 and Algorithm 1 for details. As a result, we obtain a quantized sample of size c for some $c < n$ to be stored or transmitted. We demonstrate that with a carefully chosen c and a well-designed quantizer, the proposed nonparametric estimation and testing procedures are asymptotically valid and efficient even based only on the quantized data.

Our contributions can be summarized as follows. Firstly, we propose a computationally efficient data quantization algorithm to reduce the size of the raw data to meet the B -bits constraint, and at the same time reduce the computational complexity from $O(n^3)$ to $O(c^3)$. Secondly, we establish sufficient conditions on the bits constraint, i.e., B , that warrants the minimax convergence rate for the resulting spline estimators and the minimax rates of testing for the proposed testing procedure. In particular, our results show how the asymptotic power of the proposed testing procedure changes as the bits constraint B increases. Thirdly, we further extend our theoretical investigations to (a) a nonparametric linearity test of the underlying function; (b) an adaptive nonparametric test when the smoothness of the underlying function is unknown. To the best of our knowledge, our work is the first to provide a theoretical investigation on nonparametric inference based on quantized samples.

The rest of the paper is organized as follows. Section 2 describes the general methodologies we proposed for data quantization, nonparametric estimation, and nonparametric testing using splines. In Section 3, we investigate the theoretical properties of the spline estimator and the nonparametric test statistic based on quantized samples. In Section 4, we study asymptotic properties of the nonparametric linearity test statistic and the adaptive nonparametric test statistic under B -bits constraint. Section 5 gives several simulation studies to evaluate finite sample performances of the proposed methods and Section 6 illustrates an application of the proposed methods to the Combined Cycle Power Plant Data.

Notation: Let $\|\cdot\|$ represent the L^2 -norm, i.e., $\|f\|^2 = \int_0^1 f^2(t)dt$, and define $\|\cdot\|_2$ as the Euclidean Norm of vectors. Let $\|\cdot\|_{\text{sup}}$ denote the supreme norm of a function, i.e., $\|f\|_{\text{sup}} = \sup_{t \in [0,1]} |f(t)|$. For two positive sequences a_n and b_n , we denote $a_n \gtrsim b_n$ ($a_n \lesssim b_n$) if there exists a constant $C > 0$ such that $a_n \geq Cb_n$ ($a_n \leq Cb_n$) for all $n \geq 1$;

denote $a_n \asymp b_n$ if $a_n \gtrsim b_n$ and $a_n \lesssim b_n$; denote $a_n \ll b_n$ if $a_n/b_n \rightarrow 0$ as $n \rightarrow \infty$ and $a_n \gg b_n$ if $a_n/b_n \rightarrow \infty$ as $n \rightarrow \infty$.

2. Methodology

In this section, we first review some background of the classical smoothing spline regression and then give details on the proposed quantization scheme, nonparametric estimation and testing procedures.

2.1 Review of Classical Smoothing Spline Regression

Throughout this paper, we assume that the underlying true function $g_0(\cdot)$ belongs to the m -order ($m \geq 1$) periodic Sobolev space on $\mathbb{I} := [0, 1]$ defined as

$$S^m(\mathbb{I}) = \left\{ \sum_{\nu=1}^{\infty} \beta_{\nu} \varphi_{\nu}(\cdot) : \sum_{\nu=1}^{\infty} \beta_{\nu}^2 \gamma_{\nu} < \infty \right\},$$

where $\varphi_{2k-1}(x) = \sqrt{2} \cos(2\pi kx)$, $\varphi_{2k}(x) = \sqrt{2} \sin(2\pi kx)$ are the trigonometric basis functions, and $\gamma_{2k-1} = \gamma_{2k} = (2\pi k)^{2m}$ for $x \in \mathbb{I}$ and $k \geq 1$. It follows from Wahba (1990) and Gu (2013) that $S^m(\mathbb{I})$ is a reproducing kernel Hilbert space (RKHS) endowed with an inner product $J(f, g) = \int_0^1 f^{(m)}(x)g^{(m)}(x)dx$ and a reproducing kernel

$$K(x, y) = \frac{(-1)^{m-1}}{(2m)!} B_{2m}(|x - y|),$$

where B_{2m} is the Bernoulli polynomial of order $2m$.

Based on the above assumptions on $g_0(\cdot)$, the classic smoothing spline (ss) estimator of $g_0(\cdot)$ is obtained through the following optimization problem:

$$\widehat{g}^{\text{ss}} \equiv \arg \min_{g \in S^m} \frac{1}{n} \sum_{i=1}^n (y_i - g(i/n))^2 + \lambda \int_0^1 [g^{(m)}(x)]^2 dx. \quad (3)$$

For $x \in \mathbb{I}$, we can define a function $K_x(\cdot) = K(x, \cdot)$, which belongs to $S^m(\mathbb{I})$. By the representer Theorem (Gu, 2013), the solution to (3) has the following closed-form

$$\widehat{g}^{\text{ss}} = \sum_{i=1}^n \theta_i K_{i/n}, \quad (4)$$

where $\theta = (\theta_1, \dots, \theta_n)^T = n^{-1}(\Sigma_n + \lambda I_n)^{-1}y$ with $\Sigma_n = [K(i/n, i'/n)/n]_{1 \leq i, i' \leq n} \in \mathbb{R}^{n \times n}$, $y = (y_1, \dots, y_n)^T \in \mathbb{R}^n$ and I_n being the $n \times n$ identify matrix.

To conduct hypothesis test for (2), a straightforward idea is to construct a testing statistic based on the distance between $\widehat{g}^{\text{ss}}(\cdot)$ and $g_*(\cdot)$. Specifically, we use the L_2 norm distance defined as

$$T^{\text{ss}} = \|\widehat{g}^{\text{ss}} - g_*\|^2.$$

With an appropriate normalization, it can be shown that T^{ss} is asymptotically normally distributed (Shang and Cheng, 2017; Yang et al., 2020; Liu et al., 2021).

2.2 Two-Stage Quantization

The original observations y_i 's in (1) are real-valued random variables, each of which literally requires an infinite amount of bits to store or transmit. When there are only B available bits, the original observations y_i 's may not be directly accessible for estimation or testing, and hence, the classical smoothing spline estimator given in (4) is not applicable. This section aims to introduce a two-stage quantization scheme to transform y_i 's into the ones whose storage or transmission meets the B -bits constraint. The resulting samples will be further used for optimal inferential purposes in the subsequent sections. The two-stage quantization process is demonstrated in the following Figure 1.

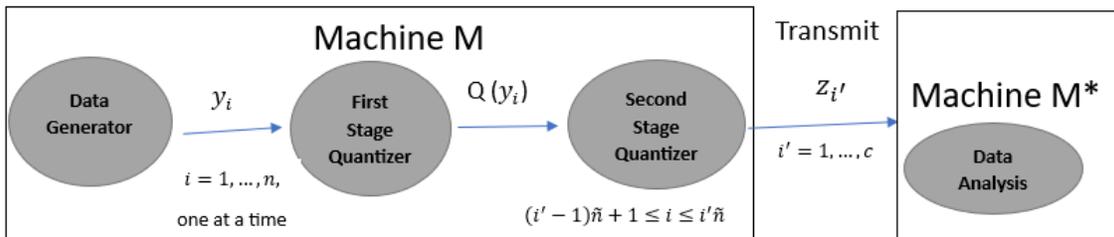


Figure 1: *Two-stage quantization process.*

The first-stage quantization is to quantize the data y_i 's as soon as they are generated with at most k distinct values. For convenience, we use a uniform quantization scheme as follows. We first choose an interval $[t_1, t_{k-1}]$ and choose $t_2 < \dots < t_{k-2}$ as the equally spaced grid points within $[t_1, t_{k-1}]$. Denote $t = (t_1, \dots, t_{k-1})^T \in \mathbb{R}^{k-1}$ and the sub-interval length $C_k(t) := (t_{k-1} - t_1)/(k - 2)$. For ease of presentation, we assume that $l_1 = t_1/C_k(t)$ is an integer. Define a quantizer $Q(\cdot)$ as follows:

$$\begin{aligned} \text{First-stage quantization:} \quad Q(y) &= \sum_{j=1}^k \mu_j I[y \in R_j(t)], \quad \text{with } \mu_j = (l_1 + j - 1)C_k(t), \\ &\text{for any } y \in \mathbb{R}, \end{aligned} \tag{5}$$

where $\mu = (\mu_1, \dots, \mu_k)^T \in \mathbb{R}^k$ consists of the quantized values and $R_1(t) = (-\infty, t_1]$, $R_2(t) = (t_1, t_2]$, \dots , $R_{k-1}(t) = (t_{k-2}, t_{k-1}]$, $R_k(t) = (t_{k-1}, \infty)$ are the corresponding quantized intervals. Clearly, the $R_j(t)$'s form a partition of the real line with assigned marks μ_j 's and Q maps each $y \in \mathbb{R}$ to one of the k marks. Applying Q to y_i 's, we generate n quantized samples $Q(y_1), \dots, Q(y_n)$, each of which takes at most k distinct values. Storage or transmission of $Q(y_i)$'s thus requires $n \log_2 k$ bits which might still go beyond the B -bits budget when $B = o(n)$. For this reason, we propose the following second-stage quantization to further reduce the storage or transmission bits through locally averaging the $Q(y_i)$'s.

The second-stage quantization is to further reduce the number of storage or transmission bits via local average. Specifically, we divide the interval $\mathbb{I} = [0, 1]$ into c equally-spaced sub-intervals for some $c \leq B/\lceil \log_2(k + 2) \rceil$. For simplicity, we assume that $\tilde{n} := n/c$ is an integer and each sub-interval contains \tilde{n} observations. The quantized data from the

first-stage quantization, i.e., $Q(y_i)$'s, are further quantized as $z_i = \tilde{l}_i C_k(t)$ such that

$$\text{Second-stage quantization: } \left| z_i - \frac{1}{\tilde{n}} \sum_{j=(i-1)\tilde{n}+1}^{i\tilde{n}} Q(y_j) \right| \leq C_k(t), \text{ for } i = 1, \dots, c. \quad (6)$$

Details of the two-stage quantization algorithm are provided in the following Algorithm 1.

Algorithm 1: Two-Stage Quantization

Initialization: set $\tilde{l}_1 = \dots = \tilde{l}_c = 0$;

Input: Data y_1, \dots, y_n that generated sequentially;

while $1 \leq i \leq c$ **do**

Set tmp = 0;

while $(i-1)\tilde{n} + 1 \leq j \leq i\tilde{n}$ **do**

(a) First-stage quantization (quantize on the spot): $Q(y_j) = l_j C_k(t)$;

(b) Second-stage quantization (averaging):

i. *update* $\tilde{l}_i = \tilde{l}_i + \text{Integer Part of } l_j / \tilde{n}$;

ii. **if** tmp + sign(l_j)($|l_j| \bmod \tilde{n}$) $\geq \tilde{n}$ **then**

update $\begin{cases} \tilde{l}_i := \tilde{l}_i + 1 \\ \text{tmp} := \text{tmp} + \text{sign}(l_j)(|l_j| \bmod \tilde{n}) - \tilde{n}; \end{cases}$

else if tmp + sign(l_j)($|l_j| \bmod \tilde{n}$) $\leq -\tilde{n}$ **then**

update $\begin{cases} \tilde{l}_i := \tilde{l}_i - 1 \\ \text{tmp} := \text{tmp} + \text{sign}(l_j)(|l_j| \bmod \tilde{n}) + \tilde{n}; \end{cases}$

else

update tmp := tmp + sign(l_j)($|l_j| \bmod \tilde{n}$);

end if

end while

end while

Output: quantized data: $z_1 = \tilde{l}_1 C_k(t), \dots, z_c = \tilde{l}_c C_k(t)$.

Based on the definition of the quantizer $Q(\cdot)$ in (5), z_i in (6) must belong to the interval $[\mu_1 - C_k(t), \mu_k + C_k(t)]$ and must be of the form $lC_k(t)$ for some integer l . Therefore, there are at most $k+2$ distinct values of z_i 's, namely, $lC_k(t)$, for $l = l_1 - 1, l_1, l_1 + 1, \dots, (l_1 + k - 1), l_1 + k$. As a result, each z_i requires $b = \lceil \log_2(k+2) \rceil$ bits to store or transmit, hence, the entire z_i 's require $c \lceil \log_2(k+2) \rceil \leq B$ bits, where $\lceil x \rceil$ is the smallest integer greater than x . In the subsequent sections, we will show that, with c, k being properly selected, optimal inferences based on z_i 's are possible even under $B = o(n)$. For optimal inferences in non-regression settings such as the Gaussian sequence model or Gaussian white-noise model, similar findings were made by Cai and Wei (2021).

We wish to remark that although Algorithm 1 focuses on the case where each sub-interval of $[0, 1]$ has the same length and contains the same number of observations, it can

be easily modified to the more general case without affecting the asymptotic properties. In particular, the simple averaging strategy can be replaced by a certain type of kernel smoothing technique.

2.3 B -bits Nonparametric Spline Estimation

Given B , let us choose k, c such that $c\lceil\log_2(k+2)\rceil = B$, i.e., our two-stage quantization maximizes the use of the available bits. Based on the quantized samples z_1, \dots, z_c from Algorithm 1, a B -bits constrained spline estimator is proposed as follows

$$\hat{g}_{\mu,t,c}^B \equiv \arg \min_{g \in S^m(\mathbb{I})} \frac{1}{c} \sum_{i=1}^c (z_i - g(i/c))^2 + \lambda \int_0^1 [g^{(m)}(x)]^2 dx. \quad (7)$$

Similar to (4), the resulting spline estimator $\hat{g}_{\mu,t,c}^B(\cdot)$ has an explicit expression

$$\hat{g}_{\mu,t,c}^B = \sum_{i=1}^c \hat{\theta}_i K_{i/c},$$

where $(\hat{\theta}_1, \dots, \hat{\theta}_c)^T = c^{-1}(\Sigma_c + \lambda I_c)^{-1}z$ with $\Sigma_c = [K(i/c, i'/c)/c]_{1 \leq i, i' \leq c} \in \mathbb{R}^{c \times c}$, $z = (z_1, \dots, z_c)^T \in \mathbb{R}^c$, and I_c being the $c \times c$ identity matrix.

Notice that the optimization of (7) only requires on c quantized observations and the solution only involves computing the inverse of a $c \times c$ matrix $\Sigma_c + \lambda I_c$, which is much less computationally intensive compared to the classical smoothing spline estimator (4).

Finally, the selection of the tuning parameter λ is crucial, and can be obtained by minimizing the generalized cross validation (GCV) score as follows

$$\hat{\lambda} = \arg \min_{\lambda > 0} \frac{c \| [I_c - \Sigma_c(\Sigma_c + \lambda I_c)^{-1}]z \|_2^2}{[c - \text{trace}(\Sigma_c(\Sigma_c + \lambda I_c)^{-1})]^2}, \quad (8)$$

The GCV has been widely used in the literature and enjoys appealing theoretical properties in various settings, see, e.g., Wahba (1990); Xu and Huang (2012); Gu (2013); Xu et al. (2018, 2019).

2.4 B -bits Nonparametric Testing

In this section, we propose a test statistic for the null hypothesis (2) based on the B -bits spline estimator $\hat{g}_{\mu,t,c}^B(\cdot)$. Without loss of generality, we assume $g_*(\cdot) \equiv 0$ in the null hypothesis (2). For a nonzero $g_*(\cdot)$, the observed response variables y_i 's from model (1) can be centered as $y_i^* = y_i - g_*(i/n)$, and the same testing procedure can be applied using y_i^* 's instead. To test $H_0 : g_0(x) = 0$, we consider test statistic based on the L_2 norm distance between $\hat{g}_{\mu,t,c}^B(\cdot)$ and $g_*(\cdot) \equiv 0$ as following

$$T_{\mu,t,c} = \|\hat{g}_{\mu,t,c}^B\|^2. \quad (9)$$

Intuitively, a large value of $T_{\mu,t,c}$ should lead to the rejection of H_0 . In Theorem 3, we shall show that under H_0 and mild conditions, it holds that

$$\frac{cT_{\mu,t,c} - \text{trace}(A)\tau_k^2}{s_c\tau_k^2} \xrightarrow{d} N(0, 1) \text{ as } n, c \rightarrow \infty,$$

where $\tau_k^2 = \text{Var}(z_1|H_0)$, $A = (\Sigma_c + \lambda I_c)^{-1} \Omega_c (\Sigma_c + \lambda I_c)^{-1}$, $\Omega_c = [K^{\otimes 2}(i/c, i'/c)/c]_{1 \leq i, i' \leq c}$ with $K^{\otimes 2}(x, x') := \int_0^1 K(x, y)K(y, x')dy$ and $s_c^2 = 2 \sum_{1 \leq i \neq i' \leq c} a_{i, i'}^2$ with $a_{i, i'}$ being the (i, i') th entry of A . In practice, τ_k^2 needs to be estimated based on the quantized data as well. We proposed the following estimator

$$\hat{\tau}_k^2 = \frac{\tilde{\tau}_n^2}{2\tilde{n}(n-1)},$$

where $\tilde{\tau}_n$ is given in the following Algorithm 2 through quantization. Intuitively, the above estimator is a re-scaled (by a factor of \tilde{n}^{-1}) version of the quantized sample error variance $\frac{1}{2(n-1)} \sum_{j=2}^n \{Q(y_j) - Q(y_{j-1})\}^2$. It is straightforward to shown that $\hat{\tau}_k^2 = \tau_k^2[1 + o_p(1)]$ under mild conditions, see Lemma 11 of Appendix C for details. Consequently, the decision rule for testing (2) at significance level α can be defined as follows

$$\phi_{c,k} = I(|cT_{\mu,t,c} - \text{trace}(A)\hat{\tau}_k^2| \geq z_{1-\alpha/2}s_c\hat{\tau}_k^2), \quad (10)$$

where $z_{1-\alpha/2}$ is the $(1 - \alpha/2)$ -percentile of the standard normal distribution. We reject the null hypothesis (2) if and only if $\phi_{c,k} = 1$.

Algorithm 2: Quantization Estimation of Variance

Initialization: set $\tilde{\tau}_n^2 = \text{tmp} = 0$;
Input: Data y_1, \dots, y_n that generated sequentially;
while $1 \leq j \leq n$ **do**
 1. Quantize on the spot $Q(y_j)$;
 2. **if** $j = 1$ **then** update $\text{tmp} := Q(y_1)$; **else**
 (a) update $\tilde{\tau}_n^2 := \tilde{\tau}_n^2 + (Q(y_j) - \text{tmp})^2$;
 (b) update $\text{tmp} := Q(y_j)$;
 end if
end while
Output: $\tilde{\tau}_n^2$.

By the design of the quantizer $Q(\cdot)$ in (5), we can see that there are at most $k+1$ distinct possible values for each $\{Q(y_{j+1}) - Q(y_j)\}^2$ ranging from $0C_k(t)^2$ to $k^2C_k(t)^2$, $1 \leq j \leq n-1$, yielding the range for $\tilde{\tau}^2$ as $[0, (n-1)k^2C_k(t)^2]$. Since $\tilde{\tau}^2$ can only take values as $lC_k(t)^2$ for some integer l , there are at most $(n-1)k^2 + 1$ distinct values for $\tilde{\tau}^2$, which would cost $\lceil \log_2 \{(n-1)k^2 + 1\} \rceil$ bits to store or transmit. Compared to the bit costs for the two-stage quantization $c \lceil \log_2(k) \rceil \approx B$, the cost to store or transmit $\tilde{\tau}^2$ is negligible when $c \rightarrow \infty$, hence is ignored in the calculation of the total bit costs for ease of presentation.

2.5 Practical Choice of c and k Given B

The implementations of Algorithms 1 and 2 require a practical choice of c and k for a given bits budget B . Based on the discussion in Section 2.2, Algorithm 1 requires $B =$

cb with $b = \lceil \log_2(k) \rceil$. Our theoretical investigations in Section 3.3 require that $b \gg \log_2 \left(\sqrt{(nh^{1/2} + n(ch)^{-1})\mathcal{T}_n} \right)$ for some $h \rightarrow 0$, and $ch \rightarrow \infty$ as $c, n \rightarrow \infty$, where \mathcal{T}_n is defined in Condition **(B)**. Furthermore, equations (16) and (17) in Section 3.3 reveal that the optimal choice of b depends on the smoothness of the periodic Sobolev space (i.e., m) and the tuning parameter λ . While the former is typically unknown in practice, the latter needs to be chosen by some data-driven criterion such as GCV based on the quantized data, which is not available until the quantization process is carried out. To simplify the calculation and make the quantization algorithm more practical, we propose to use $b = \lceil \log_2 \left(\sqrt{n\mathcal{T}_n/\sigma^2} \right) \rceil$, which is a valid choice for any m and h , and therefore is easy to use in practice. Specifically, given B , we find c and k as follows

$$\begin{aligned} c^\dagger &= \max \left\{ c \in \mathbb{Z} : c \log_2 \left(\sqrt{n\mathcal{T}_n/\sigma^2} \right) \leq B \right\}, \\ k^\dagger &= \max \left\{ k \in \mathbb{Z} : c^\dagger \lceil \log_2(k) \rceil \leq B \right\}. \end{aligned} \tag{11}$$

By the definition in Condition **(B)**, $\sqrt{\mathcal{T}_n}$ is the quantization range, and σ^2 is used in the choice of b so that \mathcal{T}_n/σ^2 is invariant if y_i 's are multiplied by a constant. Under Condition **(B)**, the actual choice of \mathcal{T}_n/σ^2 depends on the distribution of ϵ_i 's in model (1). If ϵ_i 's follow a standard Gaussian distribution, it suffices to take $\mathcal{T}_n/\sigma^2 = 2.5 \log(n)$. Therefore, σ^2 in (11) does not need to be estimated. See more discussion under Condition **(B)** regarding the choice of \mathcal{T}_n/σ^2 .

3. Asymptotic Theory

We now proceed to study asymptotic properties of the B -bits spline estimator and the nonparametric test statistic. In this section, we restrict our investigation to the simple case scenario when the order m of the periodic Sobolev space is known and fixed, and the exact form of function $g_*(\cdot)$ in the null hypothesis (2) is also known. We shall defer theoretical results on more general cases to Section 4.

3.1 Estimation Convergence Rate

We first quantify the convergence rate of $\|\hat{g}_{\mu,t,c}^B - g_0\|^2$. Even though the main focus of this paper is conducting statistical inference based on quantized samples, it is still of interest to study the asymptotic properties of the spline estimator $\hat{g}_{\mu,t,c}^B(\cdot)$. Define the Sobolev constant

$$c_s \equiv \sup_{g \in S^m(\mathbb{I})} \frac{\|g\|_{\text{sup}}}{\sqrt{J(g,g)}}. \tag{12}$$

It is known that c_s is positive finite see (Adams and Fournier, 2003).

For all our theoretical investigations, we assume that μ_j 's and t_j 's satisfy the following *boundedness* condition

Condition **(B)** : Assume that $J(g_0, g_0) \leq \rho^2$, and denote $\mathcal{T}_n = \min\{t_1^2, t_{k-1}^2\}$, it holds that,

$$\mu_j^2 P \left(|\sigma \epsilon_1| + c_s \rho > \sqrt{\mathcal{T}_n} \right) = o \left(\frac{1}{n} \right) \text{ for } j = 1, k.$$

Condition **(B)** asserts that the values of μ_1, μ_k can not be too large, and that t_1, t_k should be sufficiently large. Recall that in this paper, we adopt the uniform quantization scheme for which Condition **(B)** is rather mild. Since $J(g_0, g_0) \leq \rho^2$, by the definition of c_s in (12), we have that $\|g_0\|_{\text{sup}} \leq c_s \rho$, and we shall assume that ρ is finite for our theoretical investigation. Condition **(B)** essentially assumes that \mathcal{T}_n is sufficiently large so that all observed y_i 's fall within the quantization range with a high probability. When ϵ_i 's follow a sub-Gaussian distribution, it suffices to take $\mathcal{T}_n \asymp \log(n)$ for Condition **(B)** to hold. For distributions with heavier tails, the required order for $\min\{t_1^2, t_{k-1}^2\}$ will be larger, e.g., $\mathcal{T}_n \asymp [\log(n)]^2$ for sub-Exponential distributions. In particular, when ϵ_i 's follow a normal distribution, it suffices to use $\mathcal{T}_n = 2.5\sigma^2 \log(n)$.

Based on Condition **(B)**, the following theorem establishes an asymptotic upper bound for the estimation error $E\|\widehat{g}_{\mu,t,c}^B - g_0\|^2$.

Theorem 1 *If Condition **(B)** holds, then it follows that*

$$E\|\widehat{g}_{\mu,t,c}^B - g_0\|^2 \lesssim (nh)^{-1} + \lambda + c^{-\min\{2m,3\}} + G_{c,k}(t),$$

where $h = \lambda^{\frac{1}{2m}}$, and $G_{c,k}(t) = 4C_k(t)^2 + G_{c,k,1}(t) + G_{c,k,2}(t)$, with

$$G_{c,k,1}(t) = \frac{2}{n} \sum_{i=1}^n \int_{-\infty}^{t_1} (z - \mu_1)^2 p\left(\frac{z - g_0(i/n)}{\sigma}\right) \sigma^{-1} dz,$$

$$G_{c,k,2}(t) = \frac{2}{n} \sum_{i=1}^n \int_{t_{k-1}}^{\infty} (z - \mu_k)^2 p\left(\frac{z - g_0(i/n)}{\sigma}\right) \sigma^{-1} dz,$$

with $p(\cdot)$ being the distribution of ϵ .

The asymptotic error bound for $\widehat{g}_{\mu,t,c}^B(\cdot)$ given in Theorem 1 can be roughly categorized into three parts: (1) the estimation error of the smoothing spline estimator based on fully observed original data, i.e., $(nh)^{-1} + \lambda$ (Wahba, 1990); (2) the estimation error attributed to first-stage quantization, i.e., $G_{c,k}(t)$; and (3) the estimation bias introduced by second-stage quantization, i.e., $c^{-\min\{2m,3\}}$. An extreme case is when $t_1 \rightarrow -\infty$, $t_{k-1} \rightarrow \infty$ and $C_k(t) \rightarrow 0$, i.e., the first-stage quantizer becomes dense enough, in which case $G_{c,k}$ tends to zero, reducing to the classical nonparametric estimation setting.

Intuitively, if a sufficiently large bits budget B , and consequently sufficiently large values c and k can be used, term $(nh)^{-1} + \lambda$ will dominate the upper bound of $E\|\widehat{g}_{\mu,t,c}^B - g_0\|^2$. As a result, the convergence rate of $\|\widehat{g}_{\mu,t,c}^B - g_0\|^2$ coincides with that of the classical smoothing spline estimator based on original observations without quantization (Wahba, 1990). A sufficient condition is given in the following corollary.

Corollary 2 *Assume that Condition **(B)** holds, and that (1) $C_k(t)^2 \lesssim n^{-2m/(2m+1)}$; (2) as $T \rightarrow \infty$, $p(z)$ satisfies $\int_{|z| \geq T} z^2 p(z) dz = O(\exp(-T^d))$ where $d \geq \frac{4m}{2m+1}$; (3) $\mathcal{T}_n \asymp \log(n)$; and that (4) $c \asymp n^{\frac{\max\{1, 2m/3\}}{2m+1}}$, $\lambda \asymp n^{-\frac{2m}{2m+1}}$. Then it follows that $E\|\widehat{g}_{\mu,t,c}^B - g_0\|^2 = O(n^{-\frac{2m}{2m+1}})$, which achieves the optimal convergence rate of smoothing splines without quantization.*

Recall the definition $C_k(t) = (t_{k-1} - t_1)/(k - 2)$, under conditions of Corollary 2, the minimum order of k to achieve the optimal convergence rate is $n^{m/(2m+1)} \log(n)$, leading to a required $b = \lceil \log_2(k) \rceil \asymp \frac{m}{2m+1} \log_2(n)$. Therefore, the total bits budget $B = cb \asymp n^{\frac{\max\{1, 2m/3\}}{2m+1}} \log(n)$. Recently, Zhu and Lafferty (2018) propose a quantization scheme for the Gaussian sequence model that achieves the same optimal estimation rate with a bits budget $B \asymp n^{\frac{1}{2m+1}}$. Although their bits budget is lower than our proposed method, Zhu and Lafferty (2018) achieve this goal by essentially only quantizing the first $n^{\frac{1}{2m+1}}$ Fourier coefficients of the function $g_0(\cdot)$ and discarding the remaining Fourier coefficients as 0's. It is unclear how can this approach be extended to making valid nonparametric inferences for $g_0(\cdot)$, which is the main focus of our work. The proposed quantization scheme in Section 2.2 is in spirit closer to the quantization algorithms proposed in Slawski and Li (2018) and references therein, although these works are mainly focused on the estimation of the parametric linear regression model. In the following subsections, we shall investigate the impacts of the bits budget on the asymptotic properties of the proposed nonparametric testing procedure.

3.2 Asymptotic Distribution of the Test Statistic under H_0

In this section, we proceed to derive the asymptotic distribution of the test statistic $T_{\mu,t,c}$ under H_0 . From now on, we will use $h = \lambda^{1/(2m)}$ without repeating its definition.

Theorem 3 *Suppose that Condition (B) holds, and it holds that $h \rightarrow 0$, $ch \rightarrow \infty$, $b \gg \log_2 \left(\sqrt{(nh^{1/2} + n(ch)^{-1})\mathcal{T}_n} \right)$ and $E([\tilde{n}^{-1} \sum_{j=1}^{\tilde{n}} Q(\epsilon_j)]^4) = O(c^2 n^{-2})$ as $n, c \rightarrow \infty$. Then under H_0 , it follows that*

$$\frac{cT_{\mu,t,c} - \text{trace}(A)\hat{\tau}_k^2}{s_c \hat{\tau}_k^2} \xrightarrow{d} N(0, 1), \text{ as } n, c \rightarrow \infty, \quad (13)$$

where $T_{\mu,t,c}$, A , $\hat{\tau}_k^2$ and s_c are as defined in Section 2.4.

Theorem 3 states that under some regularity conditions, the null distribution of the nonparametric test statistic $T_{\mu,t,c}$ for H_0 in (2) is asymptotically normal. The proof relies on Stein's exchangeable pair method and is given in the Appendix.

We remark that the conditions in Theorem 3 are rather mild. Specifically, the first condition $h \rightarrow 0$ requires the tuning parameter λ to shrink to zero and the second condition $ch \rightarrow \infty$ implies the number of quantized data, ie., c , should be sufficiently large. The only condition that needs more discussion is the last condition $E([\tilde{n}^{-1} \sum_{j=1}^{\tilde{n}} Q(\epsilon_j)]^4) = O(c^2 n^{-2})$, which involves jointly controlling the moment of ϵ_i 's and the first-stage quantizer $Q(\cdot)$. Proposition 4 below provides a sufficient condition to for this assumption.

Proposition 4 *Suppose that Condition (B) holds. If $E(\epsilon_1^4) = O(nc^{-1})$, $C_k(t) = o(1)$ and $\mu_j^4 P(\sigma \epsilon_1 \in R_j(t)) = O(nc^{-1})$ for $j = 1$ and k , then it follows that $E([\tilde{n}^{-1} \sum_{j=1}^{\tilde{n}} Q(\epsilon_j)]^4) = O(c^2 n^{-2})$.*

Using Theorem 3 and Proposition 4, the validity of the proposed nonparametric testing procedure requires the quantized sample size c to be sufficiently large, in particular, $ch = c\lambda^{1/(2m)} \rightarrow \infty$. Recall that the proposed quantization scheme in Section 2.2 requires a total

bits budget $B = cb$ with $b = \lceil \log_2(k) \rceil$. As a result, for Theorem 3 to hold, the required bits budget $B \gg c \log_2 \left(\sqrt{(nh^{1/2} + n(ch)^{-1})\mathcal{T}_n} \right)$, for which the lower bound is determined by the tuning parameter λ (or h). In the next subsection, we shall investigate the impacts of λ on the asymptotic testing power against local alternatives, which can be used to study optimal asymptotic power achievable with a given bits budget B . For example, we shall show that to achieve the minimax rate of testing, one needs $B \gtrsim n^{\frac{3}{4m+1}} \log_2 \left(n^{\frac{2m}{4m+1}} \sqrt{\mathcal{T}_n} \right)$.

3.3 Asymptotic Power of the Nonparametric Test

We now proceed to examine the asymptotic power of the proposed nonparametric test. For a fixed constant $\rho > 0$, let $S_\rho^m(\mathbb{I}) = \{f \in S^m(\mathbb{I}) : J(f, f) \leq \rho^2\}$ be the ρ -ball in the periodic Sobolev space with a radius ρ . We consider the following alternative hypothesis

$$H_1 : g_0 \in S_\rho^m(\mathbb{I}) \setminus \{0\}. \quad (14)$$

Based on the definition of the quantized data z_i in (7), its unquantized counterpart can be defined as $\tilde{y}_i = \frac{1}{\tilde{n}} \sum_{j=(i-1)\tilde{n}+1}^{i\tilde{n}} y_j$ for $i = 1, \dots, c$. Under H_1 , one has that $E\tilde{y}_i = \frac{c}{\tilde{n}} \sum_{j=(i-1)\tilde{n}+1}^{i\tilde{n}} g_0(j/n)$ for $i = 1, \dots, c$. To facilitate our theoretical investigation, we introduce the following function

$$f(x) = \frac{1}{2\Delta} \int_{\max(x-\Delta, 0)}^{\min(x+\Delta, 1)} g_0(s) ds, \text{ where } x \in \mathbb{I}, \text{ and } \Delta = \frac{1}{c}. \quad (15)$$

It is straightforward to show that $\max_{1 \leq i \leq c} |f(i/c) - E\tilde{y}_i| = O(n^{-1})$ and that as $\Delta \rightarrow 0$, $\sup_{x \in \mathbb{I}} |g_0(x) - f(x)| \rightarrow 0$. Theorem 5 below states that, under some regularity conditions, our proposed nonparametric test can achieve arbitrary high power provided that H_0 and H_1 are sufficiently separated.

Theorem 5 *Suppose that Condition (B) holds. If it holds that $h \rightarrow 0$, $ch \rightarrow \infty$, $b \gg \log_2 \left(\sqrt{(nh^{1/2} + n(ch)^{-1})\mathcal{T}_n} \right)$, and $E([\tilde{n}^{-1} \sum_{j=1}^{\tilde{n}} Q(\epsilon_j)]^4) = O(c^2 n^{-2})$, then for any $\eta > 0$, there exists positive constants C_η and N_η such that for any $c \geq N_\eta$,*

$$\inf_{\substack{g \in S_\rho^m(\mathbb{I}) \\ \|g\|_c \geq C_\eta \delta_{n,c,\lambda}}} P(\text{reject } H_0 | H_1 \text{ is true}) \geq 1 - \eta,$$

where $\delta_{n,c,\lambda} = \sqrt{(nh^{1/2})^{-1} + \lambda + (nch^2)^{-1}}$ and $\|g\|_c = \sqrt{\sum_{i=1}^c f^2(i/c)/c}$ with function $f(\cdot)$ as defined in (15).

The separation rate $\delta_{n,c,\lambda}$ represents the smallest rate of deviation from the H_0 that can be consistently detected by the proposed test statistic (9), given sufficiently large n and c . The first part of $\delta_{n,c,\lambda}$, namely, $(nh^{1/2})^{-1} + \lambda$, coincides with the separation rate of the classical spline-based nonparametric test using original observations without quantization, see, e.g., Shang and Cheng (2013); Cheng and Shang (2015); Shang and Cheng (2015, 2017). The remaining part of $\delta_{n,c,\lambda}$, namely, $(nch^2)^{-1}$, is an additional term due to the two-stage quantization errors. For a given n and c , the separation rate $\delta_{n,c,\lambda}$ can be minimized

by choosing an appropriate value of the tuning parameter λ , subject to the constraint $c\lambda^{1/2m} \rightarrow \infty$. Specifically, by some straightforward algebra, one can show that

$$\inf_{\lambda \gg c^{-2m}} \delta_{n,c,\lambda} = \begin{cases} n^{-\frac{2m}{4m+1}}, & \text{if } c \gtrsim n^{\frac{3}{4m+1}} \text{ with } \lambda \asymp n^{-\frac{4m}{4m+1}}; \\ (nc)^{-\frac{m}{2(m+1)}}, & \text{if } n^{\frac{1}{2m+1}} \ll c \lesssim n^{\frac{3}{4m+1}} \text{ with } \lambda \asymp (nc)^{-\frac{m}{m+1}}; \\ \lambda^{1/2}, & \text{if } c \lesssim n^{\frac{1}{2m+1}} \text{ with } \lambda \gg c^{-2m}. \end{cases} \quad (16)$$

Recall that the total bits needed for the proposed quantization scheme in Section 2.2 is $B = cb$, for which Theorem 5 requires that $h \rightarrow 0$ and $b \gg \log_2 \left(\sqrt{(nh^{1/2} + n(ch)^{-1})\mathcal{T}_n} \right)$. By plugging the optimal smoothing parameter back to the lower bound of b , we have the following

$$\inf_{\lambda \gg c^{-2m}} \delta_{n,c,\lambda} = \begin{cases} n^{-\frac{2m}{4m+1}}, & \text{if } c \gtrsim n^{\frac{3}{4m+1}}, b \gg \log_2 \left(n^{\frac{2m}{4m+1}} \sqrt{\mathcal{T}_n} \right); \\ (nc)^{-\frac{m}{2(m+1)}}, & \text{if } n^{\frac{1}{2m+1}} \ll c \lesssim n^{\frac{3}{4m+1}}, b \gg \log_2 \left(n^{\frac{2m+3}{4(m+1)}} c^{-\frac{2m+1}{4(m+1)}} \sqrt{\mathcal{T}_n} \right); \\ \lambda^{1/2}, & \text{if } c \lesssim n^{\frac{1}{2m+1}}, b \gg \log_2 \left(\sqrt{n\mathcal{T}_n} \right) \text{ with } \lambda \gg c^{-2m}. \end{cases} \quad (17)$$

From (17), we can see that when B is sufficiently large, i.e., $B \gtrsim n^{\frac{3}{4m+1}} \log_2 \left(n^{\frac{2m}{4m+1}} \sqrt{\mathcal{T}_n} \right)$, the minimal separation rate $n^{-\frac{2m}{4m+1}}$ achieves the minimax rate of testing (Shang and Cheng, 2013, 2017; Liu et al., 2020), implying lossless asymptotic testing power using only quantized samples. In this case, the minimal number of bits for each data point, i.e., b , does not depend on c but is determined by the smoothness of the function and the tail bound \mathcal{T}_n of the error distribution. When B is between $n^{\frac{1}{2m+1}} \log_2 \left(\sqrt{n\mathcal{T}_n} \right)$ and $n^{\frac{3}{4m+1}} \log_2 \left(n^{\frac{2m}{4m+1}} \sqrt{\mathcal{T}_n} \right)$, the minimax rate of testing is no longer achievable, but the minimal separation rate still decays polynomially as the original sample size n increases. Furthermore, in this intermediate phase of B , the lower bound of b decreases as c increases, implying that increasing c rather than b when allocating the total bits budget B will more effectively improve the testing power. Finally, when B is less than $n^{\frac{1}{2m+1}} \log_2 \left(\sqrt{n\mathcal{T}_n} \right)$, the asymptotic lower bound for the minimal rate of separation is (roughly) of the order c^{-m} with $c = B/b$, the number of quantized measurements that can be transmitted or stored, provided that $b \gg \log_2 \left(\sqrt{n\mathcal{T}_n} \right)$.

4. Extensions

Our prior investigations in Section 3 assume that the hypothesized function $g_*(\cdot)$ in (2) and the order m of the periodic Sobolev space $S^m(\mathbb{I})$ are both known. In reality, it might be interesting to test other hypotheses, e.g., whether g_0 has a parametric expression such as a linear function. Meanwhile, the order m is often unknown. We will extend the prior works to such settings.

4.1 Nonparametric Testing for Linearity of $g_0(\cdot)$

In some applications, we are interested in testing whether $g_0(\cdot)$ resides in a parametric family. In this section, as an illustrative example, we consider testing the linearity of $g_0(\cdot)$:

$$H_0^{\text{linear}} : g_0 \in \mathcal{L}(\mathbb{I}) \text{ vs. } H_1^{\text{linear}} : g_0 \in S_\rho^m(\mathbb{I}) \setminus \{\mathcal{L}(\mathbb{I})\}, \quad (18)$$

where $\mathcal{L}(\mathbb{I})$ denotes the class of linear functions over $\mathbb{I} := [0, 1]$. Testing the hypothesis that $g_0(\cdot)$ belongs to other parametric families governed by a finite number of parameters can be conducted in the same way with minor modifications.

To test (18), we first obtain the least-square estimator $\hat{g}(x)$, $x \in \mathbb{I}$ based on $Q(y_j)$'s, i.e., $\hat{g}(x) = \arg \min_{g \in \mathcal{L}(\mathbb{I})} \sum_{i=1}^n [g(x_i) - Q(y_i)]^2$. Subsequently, we define the new data as $y^{\text{linear}} = (Q(y_1) - \hat{y}_1, \dots, Q(y_n) - \hat{y}_n)^T$, where $\hat{y}_i = \hat{g}(i/n)$. By applying the two-stage quantization Algorithm 1 to y^{linear} , we can then obtain the quantized data $z_{\text{linear}} = (z_{\text{linear},1}, \dots, z_{\text{linear},c})^T$. Following the same estimation procedure in Section 2.3, we can obtain a spline estimator $\hat{g}_{\text{linear},\mu,t,c}^{\text{B}}$ based on the quantized data z_{linear} .

The resulting test statistic is then defined as $T_{\text{linear},\mu,t,c} = \|\hat{g}_{\text{linear},\mu,t,c}^{\text{B}}\|^2$, whose limiting distribution under H_0^{linear} is given by the following theorem.

Theorem 6 *Suppose that Condition (B) holds. If as $n, c \rightarrow \infty$, it holds that $h \rightarrow 0$, $ch \rightarrow \infty$, $b \gg \log_2 \left(\sqrt{(nh^{1/2} + n(ch)^{-1})\mathcal{T}_n} \right)$, and $E([\tilde{n}^{-1} \sum_{j=1}^{\tilde{n}} Q(\epsilon_j)]^4) = O(c^2 n^{-2})$, then under H_0^{linear} , one has that*

$$\frac{cT_{\text{linear},\mu,t,c} - \text{trace}(A)\hat{\tau}_k^2}{s_c \hat{\tau}_k^2} \xrightarrow{d} N(0, 1), \text{ as } n, c \rightarrow \infty, \quad (19)$$

where $T_{\mu,t,c}$, A , $\hat{\tau}_k^2$ and s_c are as defined in Section 2.4 but based on y^{linear} .

Theorem 6 is an immediate extension of Theorem 3 to testing the linearity of $g_0(\cdot)$ using only quantized samples, indicating that the proposed nonparametric linearity test is valid under mild conditions. To investigate the power of the proposed linearity test against the alternative H_1^{linear} , we define the distance between $g_0(\cdot)$ and the linear function space $\mathcal{L}(\mathbb{I})$ as $\|g_0 - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g_0)\|$, where $\mathcal{P}_{\mathcal{L}(\mathbb{I})}(g_0) = \arg \min_{f \in \mathcal{L}(\mathbb{I})} \|g_0 - f\|^2$ is the projection of $g_0(\cdot)$ to $\mathcal{L}(\mathbb{I})$. The magnitude of $\|g_0 - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g_0)\|$ characterizes how far the true function $g_0(\cdot)$ deviates from any linear function in $\mathcal{L}(\mathbb{I})$. Note that under null hypothesis H_0^{linear} , one has that $\|g_0 - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g_0)\| = 0$.

The following theorem describes the asymptotic power of the proposed nonparametric linearity test.

Theorem 7 *Suppose that Condition (B) hold. If as $n, c \rightarrow \infty$, it holds that $h \rightarrow 0$, $ch \rightarrow \infty$, $b \gg \log_2 \left(\sqrt{(nh^{1/2} + n(ch)^{-1})\mathcal{T}_n} \right)$, and $E([\tilde{n}^{-1} \sum_{j=1}^{\tilde{n}} Q(\epsilon_j)]^4) = O(c^2 n^{-2})$, then for any $\eta > 0$, there exists positive constants C_η and N_η such that for any $c \geq N_\eta$,*

$$\inf_{\substack{g \in S_\rho^m(\mathbb{I}) \\ \|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c \geq C_\eta \delta_{n,c,\lambda}^{\text{linear}}}} P(\text{reject } H_0^{\text{linear}} | H_1^{\text{linear}} \text{ is true}) \geq 1 - \eta,$$

where $\delta_{n,c,\lambda}^{\text{linear}} = \sqrt{(nh^{1/2})^{-1} + \lambda + (nch^2)^{-1}}$ and $\|g\|_c = \sqrt{\sum_{i=1}^c f^2(i/c)/c}$ with function $f(\cdot)$ as defined in (15).

Based on Theorem 7, we can see that for a given quantized sample of size c , the same separation rate for testing can be achieved by the proposed nonparametric linearity test as described in (16). Furthermore, the proofs of Theorems 6-7 are similar to those of Theorem 3 and Theorem 5 by recognizing the fact that the least square estimator $\widehat{g}(\cdot)$ satisfies that $\sup_{x \in \mathbb{I}} |\widehat{g}(x) - g_0(x)| = O_p(n^{-1/2})$, whose impact is negligible for a nonparametric spline estimator. It is therefore trivial to extend Theorems 6-7 to testing whether $g_0(\cdot)$ resides in other parametric families as long as a uniformly root- n consistent parametric estimator $\widehat{g}(\cdot)$ is available.

4.2 Adaptive Nonparametric Test When m is Unknown

From (16), we can see that the power of the proposed nonparametric test depends crucially on the order m of the periodic Sobolev space where the underlying true function $g_0(\cdot)$ resides. However, the order m may be unknown in practice. One popular strategy is to set $m = 2$ regardless of the underlying truth, which may lead to sub-optimal testing power. In this section, we construct an optimal adaptive nonparametric testing procedure based on quantized samples that doesn't require m .

Let m_* denote the unknown true order of the Sobolev space to which $g_0(\cdot)$ belongs, and assume that m_* is an integer between two known integers m_l and m_u . For instance, one can set $m_l = 1$ and $m_u = \text{poly}(\log n)$ so that, as n diverges, m_* is guaranteed to belong to $[m_l, m_u]$. For any given integer m , we can calculate the test statistics $T_m := T_{\mu, t, c}$ defined in (9) with the tuning parameter $\lambda_m = a_n^{2m} n^{-4m/(4m+1)} \log(m_u)^{2m/(4m+1)}$ where a_n may depend on n but is free of m . We remark that the upper bound m_u may be slowly diverging as $n \rightarrow \infty$. Our adaptive nonparametric testing procedure is summarized as follows.

Step 1. For any $m_l \leq m \leq m_u \rightarrow \infty$, calculate the standardized testing statistic

$$\xi_m = \frac{cT_m - \text{trace}(A_m)\widehat{\tau}_k^2}{s_{c,m}\widehat{\tau}_k^2},$$

where T_m , A_m , $\widehat{\tau}_k^2$ and $s_{c,m}$ are as defined in Section 2.4.

Step 2. Calculate the maximum of ξ_m 's, i.e., $\xi_{\max} = \max_{m_l \leq m \leq m_u} \xi_m$.

Step 3. Standardize ξ_{\max} as following

$$\xi_* = C_n(\xi_{\max} - C_n),$$

where C_n satisfies $2\pi C_n^2 \exp(C_n^2) = m_u^2$.

For the validity of the proposed adaptive nonparametric test, we assume that the following Condition (C) holds.

Condition (C) : (a) $m_u \lesssim \log^{d_0}(n)$ for some $d_0 \in (0, 1/2)$, (b) $a_n n^{-2/(4m_u+1)} [\log(n)]^2 \rightarrow 0$, and (c) $n^{2/(4m_l+1)} \log(n)/(ca_n) \rightarrow 0$, as $n, c \rightarrow \infty$.

Condition (C) requires that the searching range for m can not be too large by imposing a slowly diverging upper bound on m_u . In addition, the total number of quantized samples,

i.e., c , that need to be transmitted or stored can not be too small compared to n , and is jointly determined by m_l, m_u and the tuning parameter a_n . These conditions are rather mild and have been used in the literature, see, e.g., Liu et al. (2019, 2021). The following theorem describes the asymptotic behavior of ξ_* under H_0 .

Theorem 8 *Suppose that both Conditions (B) and (C) hold, $E([\tilde{n}^{-1} \sum_{j=1}^{\tilde{n}} Q(\epsilon_j)]^4) = O(c^2 n^{-2})$, and that*

$$b \gg \max \left\{ \log_2 \left(\sqrt{(nh_{m_l}^{1/2} + n(ch_{m_l})^{-1})\mathcal{T}_n} \right), \log_2 \left(\sqrt{(nh_{m_u}^{1/2} + n(ch_{m_u})^{-1})\mathcal{T}_n} \right) \right\}.$$

Then, under H_0 given in (2), for any $\alpha \in (0, 1)$, it holds that

$$P(\xi_* \leq q_\alpha) \rightarrow 1 - \alpha, \text{ as } n, c \rightarrow \infty, \quad (20)$$

where $q_\alpha = -\log(-\log(1 - \alpha))$.

The intuition behind Theorem 8 is straightforward: under H_0 , the limiting distribution of each ξ_m is normal, which suggests that the asymptotic distribution of the maxima ξ_* should be close to the extreme value distribution. We use the techniques developed in Koike (2019) to formalize the proof.

Next, we investigate the asymptotic power of the proposed adaptive nonparametric test under the alternative $H_1 : g_0 \in S_\rho^{m_*}(\mathbb{I}) \setminus \{\mathcal{L}(\mathbb{I})\}$.

Theorem 9 *Suppose that both Conditions (B) and (C) hold, $E([\tilde{n}^{-1} \sum_{j=1}^{\tilde{n}} Q(\epsilon_j)]^4) = O(c^2 n^{-2})$, and that*

$$b \gg \max \left\{ \log_2 \left(\sqrt{(nh_{m_l}^{1/2} + n(ch_{m_l})^{-1})\mathcal{T}_n} \right), \log_2 \left(\sqrt{(nh_{m_u}^{1/2} + n(ch_{m_u})^{-1})\mathcal{T}_n} \right) \right\}.$$

Then, for any $\eta > 0$, there exists positive constants C_η and N_η such that for any $c \geq N_\eta$,

$$\inf_{\substack{g \in S_\rho^{m_*}(\mathbb{I}) \\ \|g\|_c \geq C_\eta \delta_{n,c,a_n}}} P(\text{reject } H_0 | H_1 \text{ is true}) \geq 1 - \eta,$$

where $\delta_{n,c,a_n} = n^{-\frac{2m_*}{4m_*+1}} [\log(m_u)]^{\frac{m_*}{4m_*+1}} \sqrt{a_n^{-\frac{1}{2}} + c^{-1} a_n^{-2} n^{\frac{3}{4m_*+1}} [\log(m_u)]^{-\frac{2(m_*+1)}{4m_*+1}} + a_n^{2m_*}}$ and $\|g\|_c = \sqrt{\sum_{i=1}^c f^2(i/c)/c}$ with function $f(\cdot)$ as defined in (15).

Based on the form of separation rate δ_{n,c,a_n} obtained in Theorem 9, it is straightforward to show that the minimal separation rate is obtained when $a_n = a_0$ for some constant $a_0 > 0$, provided that

$$c \gg \max\{n^{2/(4m_l+1)} \log(n), n^{3/(4m_*+1)}\} [\log(m_u)]^{-1/(4m_*+1)}$$

so that Condition (C) is met and the second term inside the square-root part of δ_{n,c,a_n} is negligible. Specifically, if $c \gg \max\{n^{2/(4m_l+1)} \log(n), n^{3/(4m_*+1)}\} [\log(m_u)]^{-1/(4m_*+1)}$, one has that

$$\inf_{a_n > 0} \delta_{n,c,a_n} \asymp n^{-\frac{2m_*}{4m_*+1}} [\log(m_u)]^{\frac{m_*}{4m_*+1}}. \quad (21)$$

The minimal separation rate (21) is the same as the one obtained in Liu et al. (2019, 2021) and is minimax for the adaptive nonparametric test. This suggests that with the quantized samples, the proposed adaptive test can still achieve the optimal testing power if the bits budget satisfies

$$B \gg \max\{n^{2/(4m_l+1)} \log(n), n^{3/(4m_*+1)}\} [\log(m_u)]^{-1/(4m_*+1)} \log(n),$$

and we take $b = \log_2 \left(\sqrt{n\mathcal{T}_n/\sigma^2} \right) \asymp \log(n)$ as suggested in Section 2.5. Compared to the minimax rate of testing when m_* is known, which is given in (16), the minimal separation rate (21) is only inflated by a factor of $[\log(m_u)]^{4m_*/(4m_*+1)}$. This is the price to pay for searching m over $m_l \leq m \leq m_u$. Furthermore, we wish to remark that the lower bound of the bits budget B depends not only on the true order m_* but also on the smallest guess of the order, i.e., m_l . This can be interpreted by the fact that ξ_{m_l} in Step 1 of the adaptive test is constructed based on an under-smoothed spline estimator, which may have a larger order of estimation bias. In practice, it is convenient to set $m_l = 1$ as suggested by Liu et al. (2021). However, a more accurate guess of m_l may lead to a smaller bits budget B required to achieve the minimax rate of testing.

5. Simulation Studies

In this section, we evaluate the finite sample performance of the proposed methods through a set of simulation studies. For all simulation settings except for Section 5.4, the data are generated from the following model

$$y_i = r\beta_{3,2}(x_i) + \epsilon_i, \quad \text{with } x_i = \frac{i}{n}, i = 1, \dots, n, \quad (22)$$

where $\beta_{3,2}(\cdot)$ is the density function of the beta distribution with parameters 3 and 2, ϵ_i 's are independent random errors. Two types of errors were considered: (1) $\epsilon \sim N(0, 1)$; (2) $\epsilon \sim N(0, 1.5^2)$. We consider r from 0 to 1, and various sample sizes n . In particular, $r = 0$ is used to examine the empirical size of the proposed test under H_0 , and other values of r are used to check the empirical powers against alternatives. The target significance level was chosen as $\alpha = 0.1$.

For all simulation studies, we consider the uniform quantization scheme outlined in Section 2.2. Specifically, for the data quantization step, for a given bits budget B , we choose c, k following the approach suggested in Section 2.5 with a $\mathcal{T}_n/\sigma^2 = 2.5 \log(n)$. For each simulation, the quantization ranges t_1, t_{k-1} are defined as $t_1 = \mu_0 - \sqrt{2.5\sigma^2 \log(n)}$, $t_{k-1} = \mu_0 + \sqrt{2.5\sigma^2 \log(n)}$, where $\mu_0 = \int_0^1 g_0(x) dx$ with $g_0(\cdot)$ being the regression function in model (1). The use of μ_0 and σ is of limited importance and can be replaced with any reasonable alternatives such as setting $\mu_0 = 0$ or using estimates based on historical data. Summary statistics from each simulation setting were based on 1000 independent simulation runs. Except for Section 5.3, we considered periodic Sobolev space of order $m = 2$ with kernel function $K(x, y) = -B_4(|x - y|)/24$, where B_4 is the Bernoulli polynomial of order 4. The tuning parameter λ was set as $\lambda = \hat{\lambda}_{\text{GCV}}/\log(c)$ with $\hat{\lambda}_{\text{GCV}}$ being picked by GCV.

5.1 Estimation Performance of $\hat{g}_{\mu,t,c}^B(\cdot)$

In this section, we first evaluate the estimation performance of the spline estimator $\hat{g}_{\mu,t,c}^B(\cdot)$ defined in (7) that is based on only quantized samples. We generated data from model (22) with $r = 0.5, 1$ and sample sizes $n = 1000, 2000, 3000, 5000, 10000$. For each n , we gradually increase the bits budget B from 30 to 1000. The estimation accuracy was evaluated by the mean squared errors (RMSE) defined as $\|\hat{g}_{\mu,t,c}^B - g_0\|$. The simulation results were summarized in Figure 2, which suggests that the MSEs decrease as n increases in all considered settings. Moreover, as B increases, the MSEs first decreases rapidly at the beginning and then stabilize at some levels. This observation is consistent with our theoretical results established in Section 3.1, which state that increasing B (or equivalently, c and b) will diminish the impact of information loss due to the data averaging and data quantization, and as a result $\hat{g}_{\mu,t,c}^B(\cdot)$ becomes more accurate. Furthermore, we can also observe after B exceeds a certain threshold, the MSEs of $\hat{g}_{\mu,t,c}^B(\cdot)$ stabilize, which supports the findings in Corollary 2. Specifically, when B is sufficiently large, the MSEs of $\hat{g}_{\mu,t,c}^B(\cdot)$ reaches the estimation error lower bound of the classical spline estimator based on the complete data.

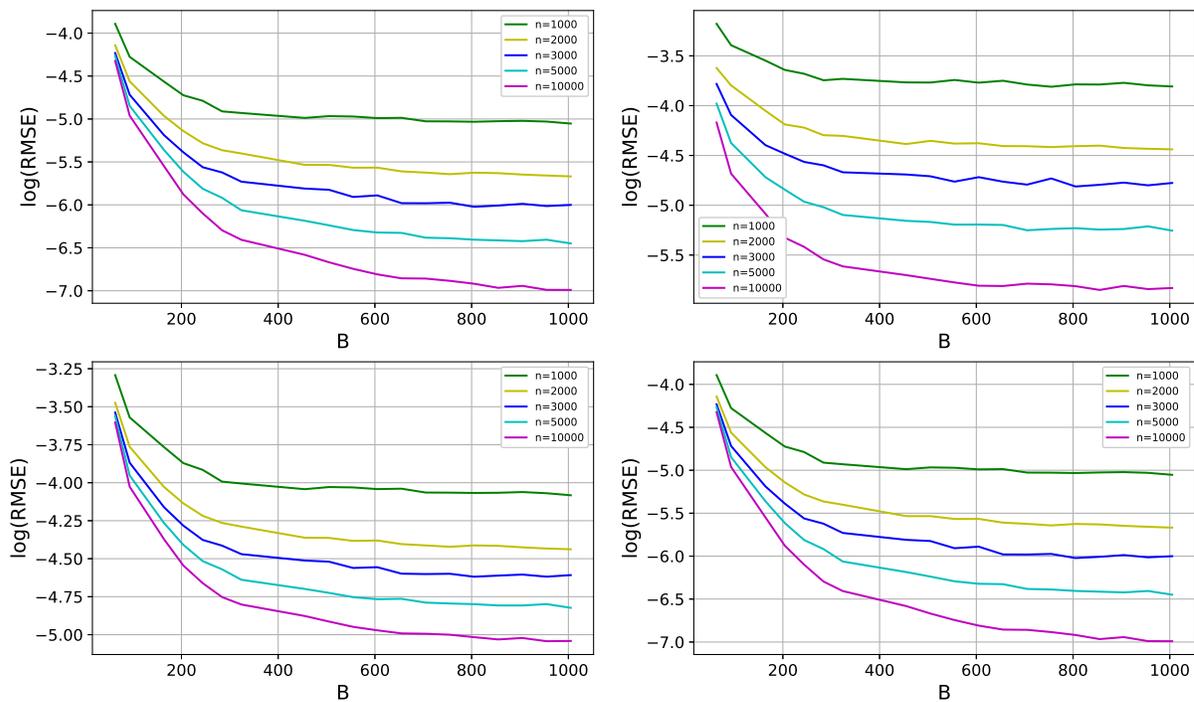


Figure 2: *MSEs of spline estimators based on quantized data: $r = 0.5$ for left 2 panels and $r = 1$ for right 2 panels; $\epsilon \sim N(0, 1)$ for top 2 panels and $\epsilon \sim N(0, 1.5^2)$ for bottom 2 panels.*

5.2 Nonparametric Test with $g_*(\cdot) \equiv 0$ and $m = 2$

In this section, we investigate the empirical sizes and powers of the nonparametric test proposed in Section 2.4, when $g_*(\cdot) \equiv 0$ in the null hypothesis (2) and $m = 2$ treated as known. The data was generated from the model (22) with various r and sample sizes n .

Figure 3 reports the empirical sizes of the proposed nonparametric test when $r = 0$ and the empirical powers when $r > 0$, respectively. Specifically, in all case scenarios, the empirical sizes of the proposed test are close to the target nominal level 0.1 as the sample size n increases. When either r or n increases, we observe that the empirical powers of the proposed test gradually approach one, which suggests that the proposed testing procedure is consistent for the alternative hypothesis that has a sufficiently large deviation (relative to the sample size n) from the H_0 . Furthermore, after the bits budget B exceeds a certain threshold, the empirical powers of the proposed nonparametric test are rather close to each other, which supports our theoretical findings in Section 3.3.

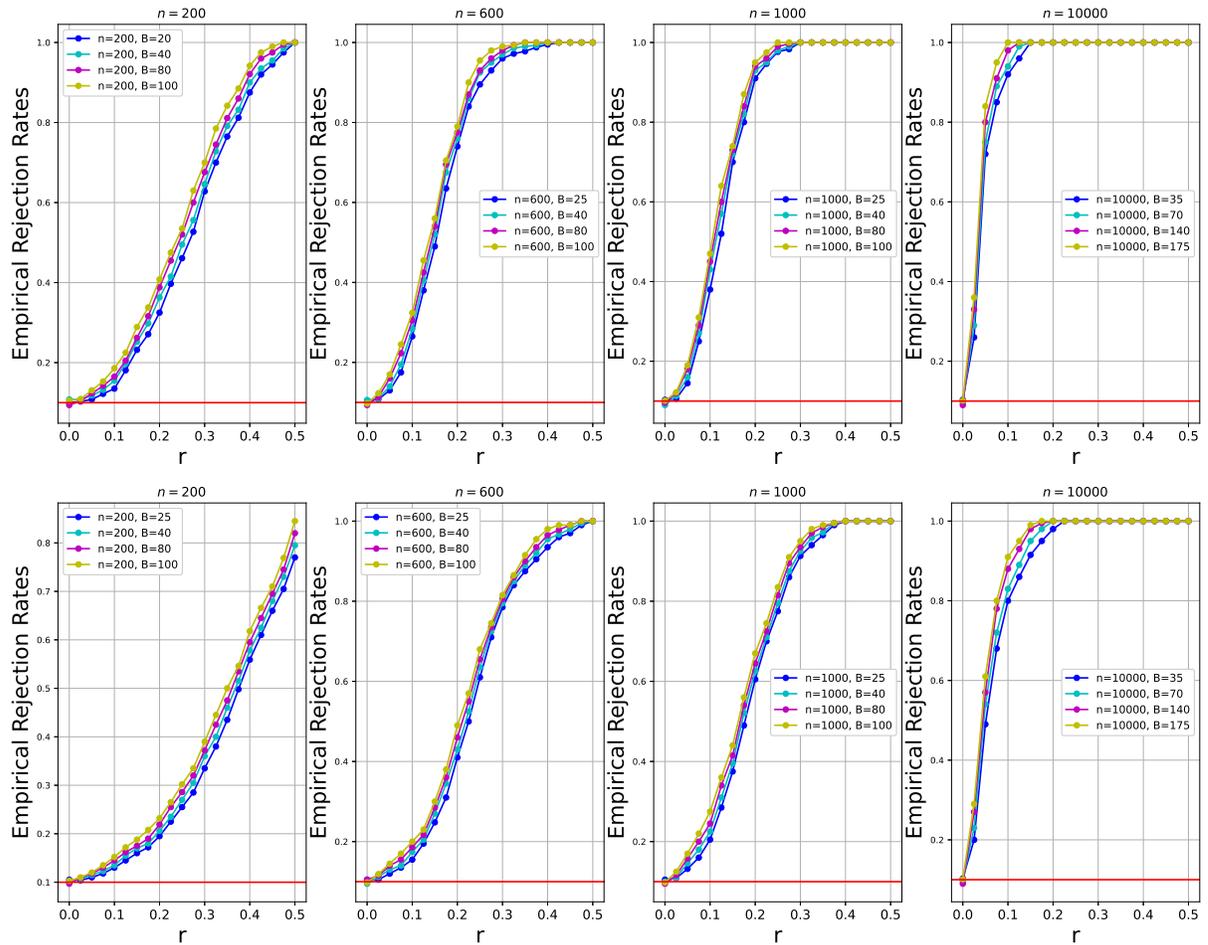


Figure 3: Empirical rejection rates at 0.1 significance level: $\epsilon \sim N(0, 1)$ for the top 4 panels; $N(0, 1.5^2)$ for the bottom 4 panels.

5.3 Adaptive Nonparametric Test with an Unknown m

In this section, we investigate the validity and the empirical power of the adaptive nonparametric test proposed in Section 4.2, for which the order parameter m is searched from $m_l = 1$ to $m_u = \sqrt{\log n}$. Figure 4 shows the empirical rejection rates of the proposed nonparametric adaptive test at the 0.1 significance level. We can observe that when $r = 0$, the empirical rejection rates are rather close to the nominal level 0.1. For any given $r > 0$, we can see that the empirical rejection rates increase as the sample size n increases. For a fixed n , as r increases, the empirical rejection rates increase steadily and eventually reach the 100% when $n = 600, n = 1000$ and $n = 10000$. Finally, as long as the bits budget B exceeds a certain threshold, the empirical rejection rates are rather similar in most settings. All these observations are consistent with our theoretical findings in Theorem 9. Furthermore, the empirical rejection rates are smaller than the nonparametric test (non-adaptive) in Section 5.2 under the same setting, which is the price paid for adaptivity in m .

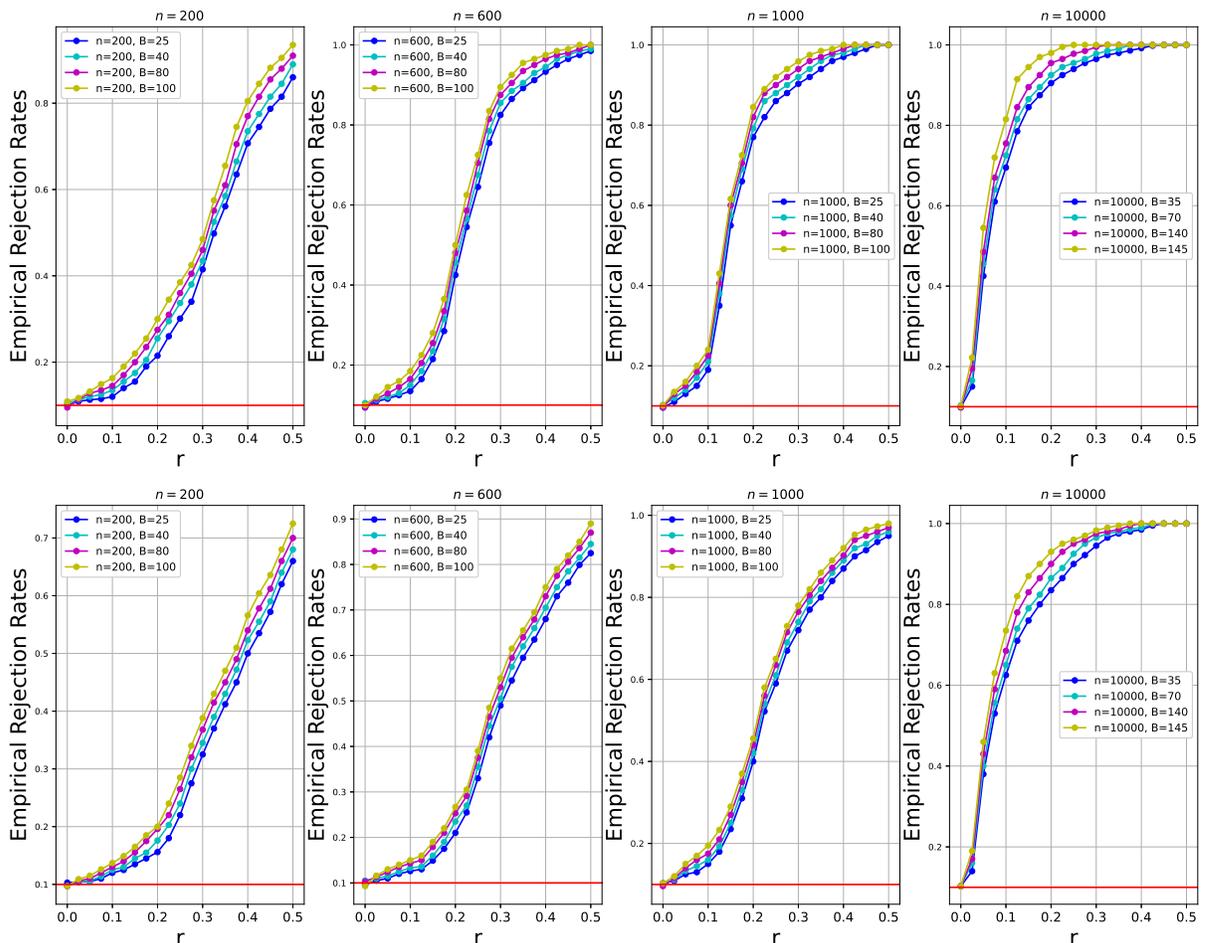


Figure 4: Empirical rejection rates of the nonparametric adaptive test at the 0.1 significance level: $\epsilon \sim N(0, 1)$ for the top 4 panels; $N(0, 1.5^2)$ for the bottom 4 panels.

5.4 Nonparametric Linearity Test with $m = 2$

In this section, we study the empirical performance of the proposed nonparametric linearity test. The data is generated from the following model

$$y_i = r\beta_{3,2}(x_i) + 3x_i + 2 + \epsilon_i, \quad \text{with } x_i = \frac{i}{n}, i = 1, \dots, n,$$

where $\beta_{3,2}(\cdot)$ is the density function of the beta distribution with parameters 3 and 2, and ϵ_i 's are independent random errors. Two types of errors were considered: (1) $\epsilon \sim N(0, 1)$; (2) $\epsilon \sim N(0, 1.5^2)$. When $r = 0$, the model satisfies the null hypothesis H_0^{linear} , and as r increases, the departure from the linear model becomes increasingly larger.

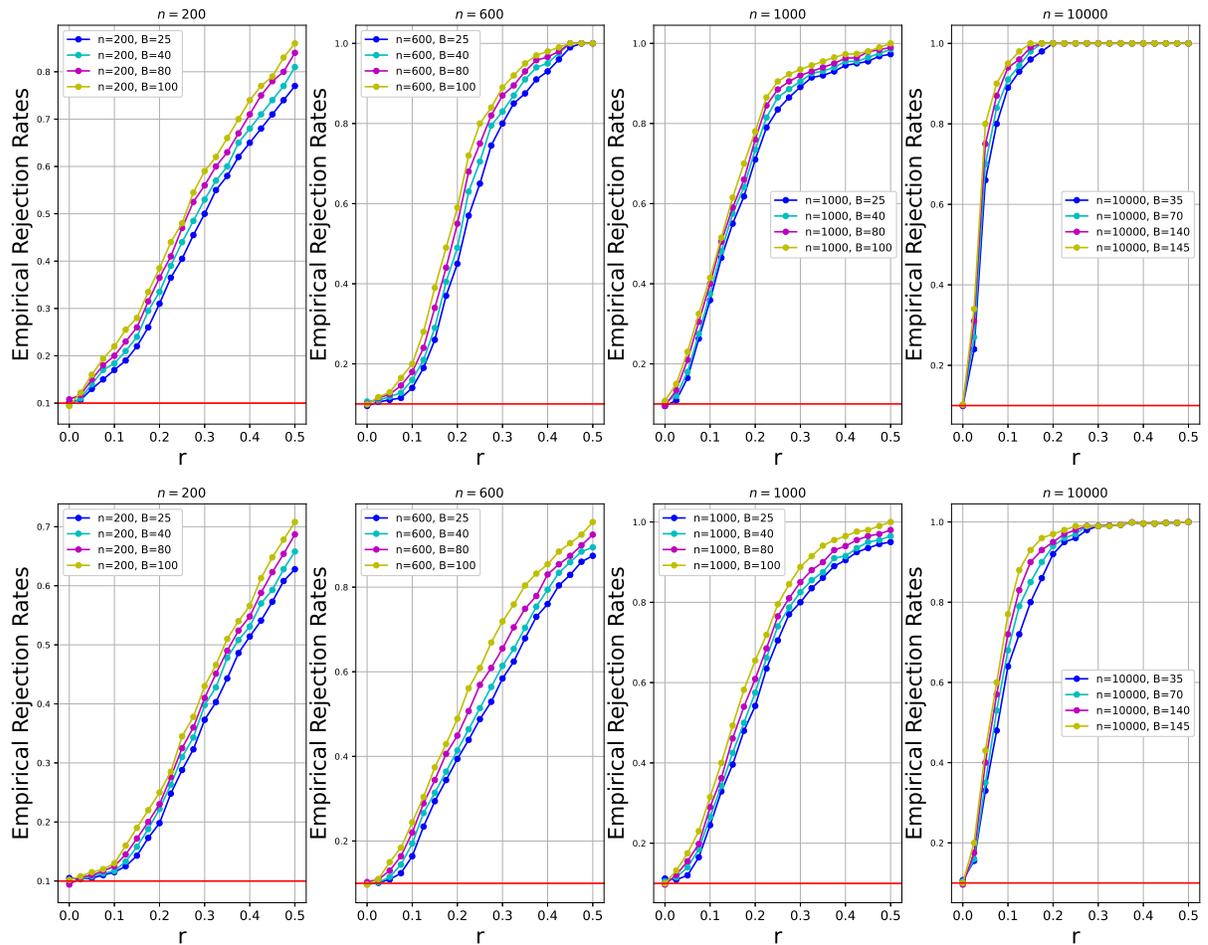


Figure 5: Empirical rejection rates of the nonparametric linearity test at 0.1 significance level: $\epsilon \sim N(0, 1)$ for the top 4 panels; $N(0, 1.5^2)$ for the bottom 4 panels.

Figure 5 reports the empirical rejection rates of the nonparametric linearity test proposed in Section (4.1) at the significance level 0.1 . It is straightforward to see that, when $r = 0$, the empirical rejection rates are close to the nominal size, indicating the validity of the test

asserted by Theorem 6. For any given $r > 0$, we can see that the empirical rejection rates increase as the sample size n increases. For a fixed n , as r increases, the empirical rejection rates increase steadily and eventually reach the 100%. Finally, as long as the bits budget B exceeds a certain threshold, the empirical rejection rates are rather similar in most settings. All these observations are consistent with our theoretical findings in Theorem 7.

6. Real Data Analysis

In this section, we apply the proposed methods to the Combined Cycle Power Plant Data (Kaya et al., 2012; Tüfekci, 2014), which can be downloaded at <http://https://archive.ics.uci.edu/ml/datasets/Combined+Cycle+Power+Plant>. The data set consists of $n = 9568$ observations from a Combined Cycle Power Plant over 6 years (2006-2011). The purpose of our analysis is to explore the relationship between the net hourly electrical energy output of the plant between three environmental factors: temperature, ambient pressure, and relative humidity.

Figure 6 displays the estimated curve based on B -bits quantizations ($B = 35, 70, 140, 175$) and full data, for which the periodic spline of order $m = 2$ was used. For the quantization step, we choose $\mathcal{T}_n = 2.5 \times \hat{\sigma}^2 \log(n)$, where $\hat{\sigma}^2$ is the standard deviation of the observed data, and c, k are determined by Section 2.5. We can observe that the spline estimator based on quantized data with $B = 35$, i.e., the green curve, is rather different from the other curves in the two analyzes. When the bits budget B increases to more than 70, such differences quickly diminish. This observation demonstrates the effectiveness of the proposed B -bits quantization scheme.

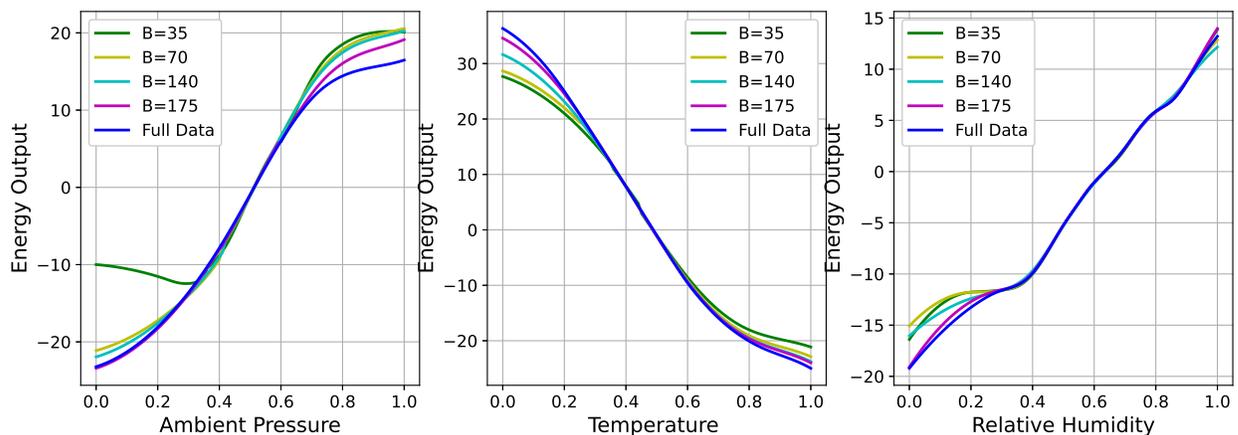


Figure 6: *Spline estimators based on B -bits quantizations ($B = 35, 70, 140, 175$) and full data. Sample size is $n = 9568$.*

Next, we conduct some hypothesis tests for the relationship between the net hourly electrical energy output and other three environmental factors. The first test is to test whether there is an association between the energy output and three environmental factors. We consider both non-adaptive and adaptive nonparametric tests. For the non-adaptive

nonparametric test, $m = 2$ is used. The p-values are all close to zero, implying strong rejections of the null hypothesis. This is not surprising based on the shapes of the spline estimators illustrated in Figure 6.

Next, there appears to be a strong linear association between relative humidity and the energy output in Figure 6. Based on this conjecture, we proceed to test whether the associations between these three environmental factors and the energy output are linear or nonlinear, using the nonparametric linearity test proposed in Section 4.1. The p-values for the first two environmental factors, i.e., ambient pressure and temperature, are both close to zero, indicating strong rejections of the null hypothesis. Figure 7 illustrates the p-values of the nonparametric linearity test for the relationship between relative humidity and energy output as a function of the bits budget B . We can see that the nonparametric linearity test based on quantized data fails to reject the null hypothesis, which echos our conjecture based on Figure 6.

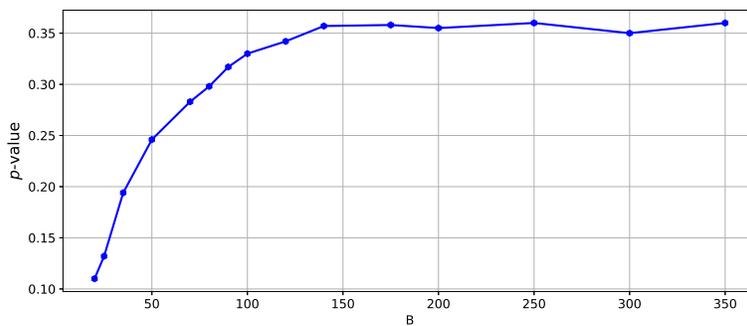


Figure 7: *P-values of the nonparametric linearity test for the relationship between relative humidity and energy output as a function of B .*

7. Discussion

In this paper, we propose a set of non-parametric testing procedures based on quantized observations, including the non-adaptive nonparametric test, the nonparametric linearity test, and the adaptive nonparametric test. The proposed tests are easy-to-use based on L_2 -metric between the quantization spline estimators and the hypothesized function. We investigate the asymptotic validity and testing powers of the proposed tests and show how the asymptotic testing powers changes as the bits budget B increases.

In the end, we discuss two additional extensions. First, the present paper only deals with periodic splines. It is interesting to extend our results to more general splines or even kernel ridge regression. The special periodic spline largely reduces the difficulty level of the technical proofs. Indeed, the majority of the proofs can be accomplished by exact calculations based on trigonometric series. For general RKHS, exact calculations may not be possible, and more involved proofs are needed. Second, the nonparametric linearity test can be easily extended to testing general composite null hypotheses such as $H_0^{\text{general}} : g_0(x) = h_0(x, \theta)$ for some function h_0 governed by parameters $\theta \in \mathbb{R}^p$ with a fixed p . However,

when p is diverging as n increases, it will be more challenging to investigate the asymptotic behavior of the proposed test statistic and will be an interesting future research topic.

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Appendix A. Structure of the proofs

In this section, we outline the high-level structure of the proofs for the main theorems.

- The proof of Theorem 1 is mainly based on Lemma 10.
 - In Lemma 10, we provide an upper bound for the difference between two smoothing spline estimators.
- The proof of Theorem 3 relies on Stein's exchangeable pair method. Specifically, we first prove that the asymptotic normality of $\frac{cT_{\mu^*,t,c} - \text{trace}(A)\tau_k^{*2}}{s_c\tau_k^{*2}}$ based on z_i^* 's, where z_i^* 's are the quantized samples corresponding to $\mu_j = \mu_j^*$ for $1 \leq j \leq k$, $\tau_k^{*2} = \text{Var}(z_i^*|H_0)$, and

$$\mu_j^* = \frac{\sum_{i=1}^n E\{y_i I(y_i \in R_j(t))\}}{\sum_{i=1}^n P(y_i \in R_j(t))}. \quad (23)$$

Next, we prove that

$$\frac{cT_{\mu^*,t,c} - \text{trace}(A)\tau_k^{*2}}{s_c\tau_k^{*2}} - \frac{cT_{\mu,t,c} - \text{trace}(A)\widehat{\tau}_k^2}{s_c\widehat{\tau}_k^2} = o_p(1).$$

- In Lemma 11 and Lemma 12, we prove the error rate introduced by quantization of variance using Algorithm 2, which are needed for the proof of Theorem 3.
- In Lemma 13, we quantify the difference of quantized sample under H_1 and H_0 .
- In the proof of Theorem 5, we first decompose the test statistic into two parts,

$$\frac{cT_{\mu,t} - \text{trace}(A)\widehat{\tau}_k^2}{s_c\widehat{\tau}_k^2} = \frac{z^T Az - (z^0)^T Az^0}{s_c\widehat{\tau}_k^2} + \frac{(z^0)^T Az^0 - \text{trace}(A)\widehat{\tau}_k^2}{s_c\widehat{\tau}_k^2},$$

where z^0 is the vector of quantized sample under $H_0 : g_0 = 0$. Under Theorem 3, we know the second term is $O_p(1)$. In the first term, it is straightforward to see that $z^T Az - (z^0)^T Az^0 = (z - z^0)^T A(z - z^0) + 2(z - z^0)^T Az^0$.

- In Lemma 14, we establish a lower bound for $(z - z^0)^T Az^0$.
- In Lemma 15 and Lemma 16, we establish the lower bound for $(z - z^0)^T A(z - z^0)$.
- In the proof of Theorem 8, observe that the test statistic for each m

$$\xi_m = \frac{cT_m - \text{trace}(A_m)\widehat{\tau}_k^2}{s_{c,m}\widehat{\tau}_k^2} = \frac{(z^0)^T A_m z^0 - \text{trace}(A_m)\widehat{\tau}_k^2}{s_{c,m}\widehat{\tau}_k^2}$$

is in a quadratic form.

- Lemma 17 proves that the maximum of the quadratic form follows an extreme value distribution.
- Lemma 18 provides the rate conditions such that Lemma 17 holds.
- The idea of Theorem 9 is similar to the proof of Theorem 5 and Theorem 7.

Appendix B. Notation

In this section, we first summarize some notations which are frequently used through out the paper for the reader's convenience.

Symbol	Description
c	number of groups.
\tilde{n}	number of observations in each group which is defined as $\tilde{n} = n/c$.
$(\mu_1, \dots, \mu_k)^T$	quantized value.
$(t_1, \dots, t_{k-1})^T$	cut-off points of quantized intervals.
$y = (y_1, \dots, y_n)^T$	vector of response .
$\tilde{y} = (\tilde{y}_1, \dots, \tilde{y}_c)^T$	average of the response which is defined as $\tilde{y}_i = \frac{1}{\tilde{n}} \sum_{j=(i-1)\tilde{n}+1}^{i\tilde{n}} y_j$.
$z = (z_1, \dots, z_c)^T$	vector of quantized sample.
$z^0 = (z_1^0, \dots, z_c^0)^T$	vector of quantized sample under $H_0 : g_0 = 0$.
$\tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_c)^T$	vector of truncated quantized sample, where $\tilde{z}_i = z_i \mathbb{1}(c_s \rho + \sigma \epsilon_j \leq \sqrt{\mathcal{T}_n})$ for all $j = (i-1)\tilde{n} + 1, \dots, i\tilde{n}$.
$\tilde{z}^0 = (\tilde{z}_1^0, \dots, \tilde{z}_c^0)^T$	vector of truncated quantized response under $H_0 : g_0 = 0$, where $\tilde{z}_i^0 = z_i^0 \mathbb{1}(c_s \rho + \sigma \epsilon_j \leq \sqrt{\mathcal{T}_n})$ for all $j = (i-1)\tilde{n} + 1, \dots, i\tilde{n}$.
$y^{\text{linear}} = (y_1^{\text{linear}}, \dots, y_n^{\text{linear}})^T$	new defined data for testing the linearity of g_0 , which is defined as $y_i^{\text{linear}} = Q(y_i) - \hat{g}(i/n)$, and $\hat{g}(i/n)$ is the least-square estimator of g .
$z^{\text{linear}} = (z_1^{\text{linear}}, \dots, z_c^{\text{linear}})^T$	vector of quantized value of y_i^{linear} .
$z_0^{\text{linear}} = (z_{i,0}^{\text{linear}}, \dots, z_{i,c}^{\text{linear}})^T$	quantized value of y_i^{linear} under $H_0^{\text{linear}} : g_0$ is linear.
λ	smoothing parameter.
$\{\varphi_i(x)\}_{i=1}^\infty$	trigonometric basis functions.
$K(\cdot, \cdot)$	kernel function.
Σ_c	kernel matrix defined as $\Sigma_c = [K(i/c, i'/c)/c]_{1 \leq i, i' \leq c}$.
Ω_c	“tensor” of $K(\cdot, \cdot)$ defined as $\Omega_c = [K^{\otimes 2}(i/c, i'/c)/c]_{1 \leq i, i' \leq c}$.
A	$A = (\Sigma_c + \lambda I_c)^{-1} \Omega_c (\Sigma_c + \lambda I_c)^{-1}$.
ζ	approximation error of Riemann sum and integral.
c_s	Sobolev constant defined as $c_s = \sup_{f \in S^m(\mathbb{I})} \frac{\ f\ _{\text{sup}}}{\sqrt{J(f)}}$.
$C_k(t)$	maximum length of quantization interval.

Table 1: Table that lists some of the useful notations that are frequently used throughout the paper.

Appendix C. Useful Lemmas

The proofs of the theorems require some preliminary lemmas. In this section, we summarize these useful lemmas. Throughout the proof, we let $\tilde{y}_i = \frac{1}{\tilde{n}} \sum_{j=(i-1)\tilde{n}+1}^{i\tilde{n}} y_j$, $i = 1, \dots, c$ and we denote \hat{g}^{ss} as the canonical smoothing spline based on the full dataset; \hat{g}_c^{ss} as the smoothing spline based on the averaged responses $\{\tilde{y}_1, \dots, \tilde{y}_c\}$, and $\hat{g}_{\mu, t, c}^{\text{B}}$ as the desired

B -bits estimator, i.e.,

$$\begin{aligned}\widehat{g}^{\text{ss}} &= \arg \min_{g \in S^m(\mathbb{I})} \frac{1}{n} \sum_{i=1}^n (y_i - g(i/n))^2 + \lambda \int_0^1 [g^{(m)}(x)]^2 dx, \\ \widehat{g}_c^{\text{ss}} &= \arg \min_{g \in S^m(\mathbb{I})} \frac{1}{c} \sum_{i=1}^c (\widetilde{y}_i - g(i/c))^2 + \lambda \int_0^1 [g^{(m)}(x)]^2 dx, \quad \widetilde{y}_i = \frac{1}{\widetilde{n}} \sum_{j=(i-1)\widetilde{n}+1}^{\widetilde{n}} y_j, \\ \widehat{g}_{\mu,t,c}^{\text{B}} &= \arg \min_{g \in S^m(\mathbb{I})} \frac{1}{c} \sum_{i=1}^c (z_i - g(i/c))^2 + \lambda \int_0^1 [g^{(m)}(x)]^2 dx.\end{aligned}$$

The following lemma describes that the distance between $\widehat{g}_{\mu,t,c}^{\text{B}}$ and $\widehat{g}_c^{\text{ss}}$ can be well controlled by carefully choosing quantization parameters μ, t and c .

Lemma 10 *For any $\mu = (\mu_1, \dots, \mu_k)^T \in \mathbb{R}^k$ and $t = (t_1, \dots, t_{k-1})^T \in \mathbb{R}^{k-1}$, it holds that*

$$\|\widehat{g}_{\mu,t,c}^{\text{B}} - \widehat{g}_c^{\text{ss}}\|^2 \leq c^{-1} \sum_{i=1}^c (z_i - \widetilde{y}_i)^2. \quad (24)$$

Proof Recall that $\widehat{g}_{\mu,t,c}^{\text{B}} = \sum_{i=1}^c \widehat{\theta}_i K_{i/c}$, where $(\widehat{\theta}_1, \dots, \widehat{\theta}_c)^T = c^{-1}(\Sigma_c + \lambda I_c)^{-1}z$ with $\Sigma_c = [K(i/c, i'/c)/c]_{1 \leq i, i' \leq c} \in \mathbb{R}^{c \times c}$, $z = (z_1, \dots, z_c)^T \in \mathbb{R}^c$, and $K(\cdot, \cdot)$ is the kernel function. Similarly, $\widehat{g}_c^{\text{ss}} = \sum_{i=1}^c \widetilde{\theta}_i K_{i/c}$, where $(\widetilde{\theta}_1, \dots, \widetilde{\theta}_c)^T = c^{-1}(\Sigma_c + \lambda I_c)^{-1}\widetilde{y}$ with $\widetilde{y} = (\widetilde{y}_1, \dots, \widetilde{y}_c)^T$. Let $\widehat{\theta} = (\widehat{\theta}_1, \dots, \widehat{\theta}_c)^T$, $\widetilde{\theta} = (\widetilde{\theta}_1, \dots, \widetilde{\theta}_c)^T$. By direct calculations, we have

$$\widehat{g}_{\mu,t,c}^{\text{B}} - \widehat{g}_c^{\text{ss}} = \sum_{\nu=1}^{\infty} \sum_{i=1}^c (\widehat{\theta}_i - \widetilde{\theta}_i) \frac{\varphi_{\nu}(i/c)}{\gamma_{\nu}} \varphi_{\nu} = \sum_{\nu=1}^{\infty} \frac{\Phi_{\nu}^T(\widehat{\theta} - \widetilde{\theta})}{\gamma_{\nu}} \varphi_{\nu},$$

where $\varphi_{2k-1}(x) = \sqrt{2} \cos(2\pi kx)$, $\varphi_{2k}(x) = \sqrt{2} \sin(2\pi kx)$ are the trigonometric basis functions, $\gamma_{2k-1} = \gamma_{2k} = (2\pi k)^{2m}$, and $\Phi_{\nu} = (\varphi_{\nu}(1/c), \varphi_{\nu}(2/c), \dots, \varphi_{\nu}(c/c))^T$. So

$$\begin{aligned}\|\widehat{g}_{\mu,t,c}^{\text{B}} - \widehat{g}_c^{\text{ss}}\|^2 &= \sum_{\nu=1}^{\infty} \frac{|\Phi_{\nu}^T(\widehat{\theta} - \widetilde{\theta})|^2}{\gamma_{\nu}^2} \\ &= (\widehat{\theta} - \widetilde{\theta})^T \sum_{\nu=1}^{\infty} \frac{\Phi_{\nu} \Phi_{\nu}^T}{\gamma_{\nu}^2} (\widehat{\theta} - \widetilde{\theta}) \\ &= c^{-1}(z - \widetilde{y})^T (\Sigma_c + \lambda I_c)^{-1} \Omega_c (\Sigma_c + \lambda I_c)^{-1} (z - \widetilde{y}).\end{aligned} \quad (25)$$

We now look at Σ_c and Ω_c . To ease our calculations, for $0 \leq l \leq c-1$, we first define the following two notations,

$$d'_l = \frac{2}{c} \sum_{k=1}^{\infty} \frac{\cos(2\pi kl/c)}{(2\pi k)^{2m}}, \quad d_l = \frac{2}{c} \sum_{k=1}^{\infty} \frac{\cos(2\pi kl/c)}{(2\pi k)^{4m}}.$$

Since $d'_l = d'_{c-l}$, $d_l = d_{c-l}$ for $l = 1, 2, \dots, c-1$, we know Σ_c, Ω_c are both symmetric circulant of order c . Furthermore, Σ_c and Ω_c share the same normalized eigenvectors as

$$x_r = \frac{1}{\sqrt{c}}(1, \varepsilon^r, \varepsilon^{2r}, \dots, \varepsilon^{(c-1)r})^T, \quad r = 0, 1, \dots, c-1,$$

where $\varepsilon = \exp(2\pi\sqrt{-1}/c)$. Let $M = (x_0, x_1, \dots, x_{c-1})$, and M^* be the conjugate transpose of M . Clearly, $MM^* = I_c$ and Σ_c, Ω_c admit the following decomposition

$$\Sigma_c = M\Lambda_{d'}M^*, \quad \Omega_c = M\Lambda_dM^*, \quad (26)$$

where $\Lambda_{d'} = \text{diag}(\lambda_{d',0}, \lambda_{d',1}, \dots, \lambda_{d',c-1})$ and $\Lambda_d = \text{diag}(\lambda_{d,0}, \lambda_{d,1}, \dots, \lambda_{d,c-1})$ with $\lambda_{d',l} = d'_0 + d'_1\varepsilon^l + \dots + d'_{c-1}\varepsilon^{(c-1)l}$ and $\lambda_{d,l} = d_0 + d_1\varepsilon^l + \dots + d_{c-1}\varepsilon^{(c-1)l}$.

Direct calculations show that

$$\lambda_{d',l} = \begin{cases} 2 \sum_{k=1}^{\infty} \frac{1}{(2\pi kc)^{2m}}, & l = 0, \\ \sum_{k=1}^{\infty} \frac{1}{[2\pi(kc-l)]^{2m}} + \sum_{k=0}^{\infty} \frac{1}{[2\pi(kc+l)]^{2m}}, & 1 \leq l \leq c-1 \end{cases}$$

$$\lambda_{d,l} = \begin{cases} 2 \sum_{k=1}^{\infty} \frac{1}{(2\pi kc)^{4m}}, & l = 0, \\ \sum_{k=1}^{\infty} \frac{1}{[2\pi(kc-l)]^{4m}} + \sum_{k=0}^{\infty} \frac{1}{[2\pi(kc+l)]^{4m}}, & 1 \leq l \leq c-1. \end{cases}$$

It is easy to examine that

$$\lambda_{d',l} = \begin{cases} 2\bar{d}'_m(2\pi c)^{-2m}, & l = 0, \\ \frac{1}{[2\pi(c-l)]^{2m}} + \frac{1}{(2\pi l)^{2m}} + \sum_{k=2}^{\infty} \frac{1}{[2\pi(kc-l)]^{2m}} + \sum_{k=1}^{\infty} \frac{1}{[2\pi(kc+l)]^{2m}}, & 1 \leq l \leq c-1, \end{cases} \quad (27)$$

$$\lambda_{d,l} = \begin{cases} 2\bar{d}_m(2\pi c)^{-4m}, & l = 0, \\ \frac{1}{[2\pi(c-l)]^{4m}} + \frac{1}{(2\pi l)^{4m}} + \sum_{k=2}^{\infty} \frac{1}{[2\pi(kc-l)]^{4m}} + \sum_{k=1}^{\infty} \frac{1}{[2\pi(kc+l)]^{4m}}, & 1 \leq l \leq c-1, \end{cases} \quad (28)$$

$$\underline{d}'_m(2\pi c)^{-2m} \leq \sum_{k=2}^{\infty} \frac{1}{[2\pi(kc-l)]^{2m}} \leq \bar{d}'_m(2\pi c)^{-2m},$$

$$\underline{d}'_m(2\pi c)^{-2m} \leq \sum_{k=1}^{\infty} \frac{1}{[2\pi(kc+l)]^{2m}} \leq \bar{d}'_m(2\pi c)^{-2m},$$

$$\underline{d}_m(2\pi c)^{-4m} \leq \sum_{k=2}^{\infty} \frac{1}{[2\pi(kc-l)]^{4m}} \leq \bar{d}_m(2\pi c)^{-4m},$$

$$\underline{d}_m(2\pi c)^{-4m} \leq \sum_{k=1}^{\infty} \frac{1}{[2\pi(kc+l)]^{4m}} \leq \bar{d}_m(2\pi c)^{-4m},$$

where

$$\bar{d}'_m := \sum_{k=1}^{\infty} k^{-2m}, \quad \underline{d}'_m := \sum_{k=2}^{\infty} k^{-2m}, \quad \bar{d}_m := \sum_{k=1}^{\infty} k^{-4m}, \quad \underline{d}_m := \sum_{k=2}^{\infty} k^{-4m}. \quad (29)$$

It follows from (27) and (28) that $\lambda_{d,l} \leq \lambda_{d',l}^2$ for $0 \leq l \leq c-1$. Therefore,

$$\begin{aligned} & (z - \tilde{y})^T (\Sigma_c + \lambda I_c)^{-1} \Omega_c (\Sigma_c + \lambda I_c)^{-1} (z - \tilde{y}) \\ &= (z - \tilde{y})^T M \text{diag} \left(\frac{\lambda_{d,0}}{(\lambda + \lambda_{d',0})^2}, \dots, \frac{\lambda_{d,c-1}}{(\lambda + \lambda_{d',c-1})^2} \right) M^* (z - \tilde{y}) \\ &\leq (z - \tilde{y})^T M M^* (z - \tilde{y}) = (z - \tilde{y})^T (z - \tilde{y}). \end{aligned}$$

Therefore, it follows by (25) that

$$\|\widehat{g}_{\mu,t,c}^B - \widehat{g}_c^{ss}\|^2 \leq c^{-1}(z - \widetilde{y})^T(z - \widetilde{y}) = c^{-1} \sum_{i=1}^c (z_i - \widetilde{y}_i)^2. \quad (30)$$

This completes the proof. \blacksquare

Lemma 11 *Suppose Condition (B) holds true, and it holds that $C_k(t) \rightarrow 0$, then we have $\tau_k^2 = \text{Var}(z_1|H_0) = O(\widetilde{n}^{-1} + C_k(t)^2)$ and $\widehat{\tau}_k^2 = \frac{\widetilde{\tau}_n^2}{2\widetilde{n}(n-1)} = \tau_k^2 [1 + O_p(n^{-1/2} + C_k(t))] = \tau_k^2 [1 + o_p(1)]$.*

Proof By the definition of τ_k^2 and (6) we have

$$\begin{aligned} \tau_k^2 &= \text{Var}(z_1|H_0) = \frac{1}{\widetilde{n}^2} \text{Var}\left(\sum_{i=1}^{\widetilde{n}} Q(y_i)|H_0\right) + O(C_k(t)^2) = \frac{1}{\widetilde{n}} \text{Var}(Q(\sigma\epsilon_1)) + O(C_k(t)^2) \\ &= \frac{1}{\widetilde{n}} \sum_{j=1}^k \mu_j^2 P(Q(\sigma\epsilon_1) = \mu_j) - \frac{1}{\widetilde{n}} \left(\sum_{j=1}^k \mu_j P(Q(\sigma\epsilon_1) = \mu_j) \right)^2 + O(C_k(t)^2) \\ &= \frac{1}{\widetilde{n}} \sum_{j=1}^k \mu_j^2 P(\sigma\epsilon_1 \in R_j(t)) - \frac{1}{\widetilde{n}} \left(\sum_{j=1}^k \mu_j P(\sigma\epsilon_1 \in R_j(t)) \right)^2 + O(C_k(t)^2) \\ &= R_1 + R_2 + O(C_k(t)^2). \end{aligned} \quad (31)$$

Assume that for $2 \leq s \leq k-1$, $t_1 < t_2 < \dots < t_{s-1} \leq 0 < t_s < \dots < t_{k-1}$ and let $p(\epsilon)$ be the density function of ϵ_1 . Then we have

$$\begin{aligned} \sum_{j=2}^{s-1} \mu_j^2 P(\sigma\epsilon_1 \in R_j(t)) &\leq \sum_{j=2}^{s-1} \int_{t_{j-1}/\sigma}^{t_j/\sigma} p(\epsilon) d\epsilon t_{j-1}^2 \\ &\leq 2 \sum_{j=2}^{s-1} \int_{t_{j-1}/\sigma}^{t_j/\sigma} p(\epsilon) d\epsilon (t_j^2 + C_k(t)^2) \\ &\leq 2\sigma^2 \sum_{j=2}^{s-1} \int_{t_{j-1}/\sigma}^{t_j/\sigma} \epsilon^2 p(\epsilon) d\epsilon + 2C_k(t)^2 \sum_{j=2}^{s-1} \int_{t_{j-1}/\sigma}^{t_j/\sigma} p(\epsilon) d\epsilon \end{aligned}$$

and

$$\begin{aligned} \sum_{j=s+1}^{k-1} \mu_j^2 P(\sigma\epsilon_1 \in R_j(t)) &\leq \sum_{j=s+1}^{k-1} \int_{t_{j-1}/\sigma}^{t_j/\sigma} p(\epsilon) d\epsilon t_j^2 \\ &\leq 2 \sum_{j=s+1}^{k-1} \int_{t_{j-1}/\sigma}^{t_j/\sigma} p(\epsilon) d\epsilon (t_{j-1}^2 + C_k(t)^2) \\ &\leq 2\sigma^2 \sum_{j=s+1}^{k-1} \int_{t_{j-1}/\sigma}^{t_j/\sigma} \epsilon^2 p(\epsilon) d\epsilon + 2C_k(t)^2 \sum_{j=s+1}^{k-1} \int_{t_{j-1}/\sigma}^{t_j/\sigma} p(\epsilon) d\epsilon. \end{aligned}$$

The fact that $|\mu_s| \leq C_k(t)$ and the above inequalities lead

$$\begin{aligned} R_1 &= \frac{1}{\tilde{n}} \left\{ \sum_{j=2}^{s-1} \mu_j^2 P(\sigma\epsilon_1 \in R_j(t)) + \sum_{j=s+1}^{k-1} \mu_j^2 P(\sigma\epsilon_1 \in R_j(t)) \right. \\ &\quad \left. + \mu_1^2 P(\sigma\epsilon_1 \in R_1(t)) + \mu_k^2 P(\sigma\epsilon_1 \in R_k(t)) + \mu_s^2 P(\sigma\epsilon_1 \in R_s(t)) \right\} \\ &\leq \frac{1}{\tilde{n}} \left\{ 2\sigma^2 \int_{\mathbb{R}} \epsilon^2 p(\epsilon) d\epsilon + 3C_k(t)^2 + O(1) \right\}. \end{aligned}$$

This proves $R_1 \lesssim \frac{1}{\tilde{n}}$. On the other hand, by $t_1^2 > \sigma^2$ and $t_{s-1} = O(C_k(t)) = o(1)$, we have

$$\begin{aligned} R_1 &\geq \frac{1}{\tilde{n}} \sum_{j=2}^{s-1} \mu_j^2 P(\sigma\epsilon_1 \in R_j(t)) \\ &\geq \frac{1}{\tilde{n}} \sum_{j=2}^{s-1} t_j^2 \int_{t_{j-1}/\sigma}^{t_j/\sigma} p(\epsilon) d\epsilon \\ &\geq \frac{1}{\tilde{n}} \sum_{j=2}^{s-1} (t_{j-1}^2/2 - C_k(t)^2) \int_{t_{j-1}/\sigma}^{t_j/\sigma} p(\epsilon) d\epsilon \\ &\geq \frac{1}{\tilde{n}} \left\{ \frac{\sigma^2}{2} \sum_{j=2}^{s-1} \int_{t_{j-1}/\sigma}^{t_j/\sigma} \epsilon^2 p(\epsilon) d\epsilon - C_k(t)^2 \sum_{j=2}^{s-1} \int_{t_{j-1}/\sigma}^{t_j/\sigma} p(\epsilon) d\epsilon \right\} \\ &\geq \frac{1}{\tilde{n}} \left\{ \frac{\sigma^2}{2} \int_{t_1/\sigma}^{t_{s-1}/\sigma} \epsilon^2 p(\epsilon) d\epsilon - C_k(t)^2 \right\} = O(\tilde{n}^{-1}). \end{aligned}$$

This proves $R_1 \gtrsim \frac{1}{\tilde{n}}$, which implies $R_1 = O(\tilde{n}^{-1})$. Using a similar approach, we can prove $R_2 = O(\tilde{n}^{-1})$. From (31), we get $\tau_k^2 = O(\tilde{n}^{-1} + C_k(t)^2)$.

Now we prove $\hat{\tau}_k^2 = \frac{\tilde{\tau}_n^2}{2\tilde{n}(n-1)} = \tau_k^2[1 + o_p(1)]$. By the definition of $\tilde{\tau}_n$, we have

$$\begin{aligned} \frac{\tilde{\tau}_n^2}{2\tilde{n}(n-1)} &= \frac{1}{2\tilde{n}(n-1)} \sum_{i=2}^n (Q(y_i) - Q(y_{i-1}))^2 \\ &= \frac{1}{2\tilde{n}(n-1)} \sum_{i=2}^n \{Q(g_0(i/n) + \sigma\epsilon_i) - Q(g_0((i-1)/n) + \sigma\epsilon_{i-1})\}^2 \\ &= \frac{1}{2\tilde{n}(n-1)} \sum_{i=2}^n \left\{ g_0(i/n) - g_0((i-1)/n) + Q(\sigma\epsilon_i) - Q(\sigma\epsilon_{i-1}) + \tilde{\delta}_i \right\}^2 \\ &= \frac{1}{2\tilde{n}(n-1)} \sum_{i=2}^n \{Q(\sigma\epsilon_i) - Q(\sigma\epsilon_{i-1})\}^2 + \frac{1}{2\tilde{n}(n-1)} \sum_{i=2}^n \left\{ g_0(i/n) - g_0((i-1)/n) + \tilde{\delta}_i \right\}^2 \\ &\quad + \frac{1}{\tilde{n}(n-1)} \sum_{i=2}^n \{Q(\sigma\epsilon_i) - Q(\sigma\epsilon_{i-1})\} \left\{ g_0(i/n) - g_0((i-1)/n) + \tilde{\delta}_i \right\}, \end{aligned}$$

where $|\tilde{\delta}_i| = O(C_k(t))$ for $i = 1, \dots, n$. Note that, by the central limit theorem, it holds that

$$\frac{1}{2(n-1)} \sum_{i=2}^n \{Q(\sigma\epsilon_i) - Q(\sigma\epsilon_{i-1})\}^2 = \text{Var}[Q(\sigma\epsilon_1)] + O_p(n^{-1/2}),$$

and that $g_0(i/n) - g_0((i-1)/n) = O(1/n)$ by the smoothness of function g_0 , we have that

$$\begin{aligned} \frac{\tilde{\tau}_n^2}{2\tilde{n}(n-1)} &= \frac{1}{\tilde{n}} \left[\text{Var}(Q(\sigma\epsilon_1)) + O(n^{-2} + C_k(t)^2) + O_p(n^{-1/2}) + O(n^{-1} + C_k(t)) \right] \\ &= \tau_k^2 \left[1 + O_p(n^{-1/2} + C_k(t)) \right], \end{aligned} \quad (32)$$

which completes the proof. \blacksquare

Lemma 12 *Suppose Condition (B) holds true, and it holds that $C_k(t) \rightarrow 0$. Let $\hat{\tau}_k^2$ be the quantied variance based on y^{linear} , then we have that $\hat{\tau}_k^2 = \frac{\tilde{\tau}_n^2}{2\tilde{n}(n-1)} = \tau_k^2 [1 + O_p(n^{-1/2} + C_k(t))] = \tau_k^2 [1 + o_p(1)]$.*

Proof By the definition of $\hat{\tau}_k^2$, we have

$$\begin{aligned} \frac{\tilde{\tau}_n^2}{2\tilde{n}(n-1)} &= \frac{1}{2\tilde{n}(n-1)} \sum_{i=2}^n (Q(y_i^{\text{linear}}) - Q(y_{i-1}^{\text{linear}}))^2 \\ &= \frac{1}{2\tilde{n}(n-1)} \sum_{i=2}^n \{Q[Q(g_0(i/n) + \sigma\epsilon_i) - \hat{y}_i] - Q[Q(g_0((i-1)/n) + \sigma\epsilon_{i-1}) - \hat{y}_{i-1}]\}^2 \\ &= \frac{1}{2\tilde{n}(n-1)} \sum_{i=2}^n \left\{ g_0(i/n) - g_0((i-1)/n) + \hat{y}_i - \hat{y}_{i-1} + Q(\sigma\epsilon_i) - Q(\sigma\epsilon_{i-1}) + \tilde{\delta}_i \right\}^2, \end{aligned}$$

where $|\tilde{\delta}_i| = O(C_k(t))$ for $i = 1, \dots, n$. Note that $\hat{g} \in \mathcal{L}(\mathbb{I})$, one has that $|\hat{y}_i - \hat{y}_{i-1}| = O_p(1/n)$ by the smoothness of \hat{g} . Similar to the proof in Lemma 11, we get $\hat{\tau}_k^2 = \frac{\tilde{\tau}_n^2}{2\tilde{n}(n-1)} = \tau_k^2 [1 + O_p(n^{-1/2} + C_k(t))] = \tau_k^2 [1 + o_p(1)]$. \blacksquare

To ease calculation, we define some useful notations. Let z_i^0 be the quantized data conditional on $g_0(x) = 0$ and $z^0 = (z_1^0, \dots, z_c^0)^T$. According to (6), we have

$$\left| z_i^0 - \frac{1}{\tilde{n}} \sum_{j=(i-1)\tilde{n}+1}^{i\tilde{n}} Q(\sigma\epsilon_j) \right| \leq C_k(t), \quad \text{for } i = 1, \dots, c. \quad (33)$$

Furthermore, we let $\tilde{z}^0 = (\tilde{z}_1^0, \dots, \tilde{z}_c^0)^T$, $\tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_c)^T$, where for $i = 1, \dots, c$,

$$\begin{aligned} \tilde{z}_i^0 &= z_i^0 \mathbb{1}(c_s \rho + \sigma|\epsilon_j| \leq \sqrt{\mathcal{T}_n}) \text{ for all } j = (i-1)\tilde{n} + 1, \dots, i\tilde{n}, \\ \tilde{z}_i &= z_i \mathbb{1}(c_s \rho + \sigma|\epsilon_j| \leq \sqrt{\mathcal{T}_n}) \text{ for all } j = (i-1)\tilde{n} + 1, \dots, i\tilde{n}. \end{aligned}$$

Lemma 13 *Suppose g is the regression function generating the samples. Suppose Condition (B) holds and $\sigma|\epsilon_j| + c_s\rho \leq \sqrt{\mathcal{T}_n}$ holds for all $j = 1, \dots, n$. Then for any $g \in S^m(\mathbb{I})$ with $J(g) \leq \rho^2$, it holds that $|\tilde{z}_i - \tilde{z}_i^0 - f(i/c)| \leq 4C_k(t) + \zeta, i = 1, \dots, c$, where f is the corresponding integral equation defined in (15), $\zeta = \max_{i=1, \dots, c} |f(i/c) - \frac{1}{\tilde{n}} \sum_{j=(i-1)\tilde{n}+1}^{\tilde{n}} g(j/n)| = \max_{i=1, \dots, c} |\frac{1}{2\Delta} \int_{\max(i/c-\Delta, 0)}^{\min(i/c+\Delta, 1)} g(s)ds - \frac{1}{\tilde{n}} \sum_{j=(i-1)\tilde{n}+1}^{\tilde{n}} g(j/n)|$ and $\Delta = \frac{1}{c}$.*

Proof: Suppose $\sigma\epsilon_i \in R_j(t)$ for some $1 \leq j \leq k$. Since $\min\{t_1^2, t_{k-1}^2\} = \mathcal{T}_n$ and $c_s\rho + \sigma|\epsilon_i| \leq \sqrt{\mathcal{T}_n}$, we must have $2 \leq j \leq k-1$. Suppose that $g(i/n) + \sigma\epsilon_i \in R_l(t)$ for some $1 \leq l \leq k$. Since $\min\{t_1^2, t_{k-1}^2\} = \mathcal{T}_n$ and by (12) implying $|g(i/n)| \leq c_s\rho$, we have

$$|y_i| = |g(i/n) + \sigma\epsilon_i| \leq |g(i/n)| + |\sigma\epsilon_i| \leq c_s\rho + |\sigma\epsilon_i| \leq \sqrt{\mathcal{T}_n} = \min\{|t_1|, |t_{k-1}|\}.$$

Therefore, $2 \leq l \leq k-1$. Since

$$t_{j-1} \leq \sigma\epsilon_i < t_j, \quad t_{l-1} \leq g(i/n) + \sigma\epsilon_i < t_l,$$

$$t_{j-1} \leq \mu_j < t_j, \quad t_{l-1} \leq \mu_l < t_l,$$

we have

$$t_{l-1} - t_j < \mu_l - \mu_j < t_l - t_{j-1},$$

$$t_{l-1} - t_j < g(i/n) < t_l - t_{j-1}.$$

Hence it holds that

$$|Q(y_i) - Q(\sigma\epsilon_i) - g(i/n)| = |\mu_l - \mu_j - g(i/n)| \leq |t_l - t_{l-1}| + |t_j - t_{j-1}| \leq 2C_k(t). \quad (34)$$

Since $c_s\rho + \sigma|\epsilon_j| \leq \sqrt{\mathcal{T}_n}$ for all $j = 1, \dots, n$, the result follows from (6) and (33) that

$$\begin{aligned} |\tilde{z}_i - \tilde{z}_i^0 - f(i/c)| &= \left| z_i - \frac{\sum_{j=(i-1)\tilde{n}+1}^{\tilde{n}} Q(y_j)}{\tilde{n}} - \left(z_i^0 - \frac{\sum_{j=(i-1)\tilde{n}+1}^{\tilde{n}} Q(\sigma\epsilon_j)}{\tilde{n}} \right) \right. \\ &\quad \left. + \frac{\sum_{j=(i-1)\tilde{n}+1}^{\tilde{n}} Q(y_j)}{\tilde{n}} - \frac{\sum_{j=(i-1)\tilde{n}+1}^{\tilde{n}} Q(\sigma\epsilon_j)}{\tilde{n}} - f(i/c) \right| \\ &\leq \frac{\left| \sum_{j=(i-1)\tilde{n}+1}^{\tilde{n}} ((Q(y_j) - Q(\sigma\epsilon_j)) - \tilde{n}f(i/c)) \right|}{\tilde{n}} + 2C_k(t) \\ &= \frac{\left| \sum_{j=(i-1)\tilde{n}+1}^{\tilde{n}} \{Q(y_j) - Q(\sigma\epsilon_j) - g(j/n)\} + \sum_{j=(i-1)\tilde{n}+1}^{\tilde{n}} g(j/n) - \tilde{n}f(i/c) \right|}{\tilde{n}} + 2C_k(t) \\ &\leq \frac{\left| \sum_{j=(i-1)\tilde{n}+1}^{\tilde{n}} \{Q(y_j) - Q(\sigma\epsilon_j) - g(j/n)\} \right| + \left| \sum_{j=(i-1)\tilde{n}+1}^{\tilde{n}} g(j/n) - \tilde{n}f(i/c) \right|}{\tilde{n}} + 2C_k(t) \\ &\leq 4C_k(t) + \zeta, \end{aligned}$$

where the last inequality follows from (34) and the definition of ζ . ■

Lemma 14 *Suppose Condition (B) holds, and $h \rightarrow 0$, $ch \rightarrow \infty$. Then for any $g \in S^m(\mathbb{I})$ with $J(g) \leq \rho^2$, we have*

$$\sup_{g \in S^m(\mathbb{I})} \frac{E\{|\tilde{z} - \tilde{z}^0\}^T A z^0|^2\}}{(1 + (ch^2)^{-1})(\tau_k^2 + 4C_k(t)^2) \sum_{i=1}^c (|f(i/c)| + 4C_k(t) + \zeta)^2} \leq 8, \quad \text{as } c \rightarrow \infty, \quad (35)$$

where $\zeta = \max_{i=1, \dots, c} \left| \frac{1}{2\Delta} \int_{\max(i/c-\Delta, 0)}^{\min(i/c+\Delta, 1)} g(s) ds - \frac{1}{\tilde{n}} \sum_{j=(i-1)\tilde{n}+1}^{\tilde{i}\tilde{n}} g(j/n) \right|$ and $\Delta = \frac{1}{c}$.

Proof: For convenience, let $\omega_i = \tilde{z}_i - \tilde{z}_i^0$. From Lemma 13 and the fact that $\tilde{z}_i - \tilde{z}_i^0 = 0$ if $c_s \rho + \sigma |\epsilon_j| > \sqrt{T_n}$ for some $(i-1)\tilde{n} + 1 \leq j \leq \tilde{i}\tilde{n}$, it holds that

$$|\omega_i| \leq |f(i/c)| + 4C_k(t) + \zeta \quad \text{for all } 1 \leq i \leq c. \quad (36)$$

According to (33) and the fact that

$$\left| E \left(\frac{1}{\tilde{n}} \sum_{j=(i-1)\tilde{n}+1}^{\tilde{i}\tilde{n}} Q(\sigma \epsilon_j) \right) \right| \leq E \left(\frac{1}{\tilde{n}} \sum_{j=(i-1)\tilde{n}+1}^{\tilde{i}\tilde{n}} \sigma \epsilon_j \right) + C_k(t),$$

one has that

$$|E(z_i^0)| \leq \left| E \left(\frac{1}{\tilde{n}} \sum_{j=(i-1)\tilde{n}+1}^{\tilde{i}\tilde{n}} Q(\sigma \epsilon_j) \right) \right| + C_k(t) \leq 2C_k(t),$$

which further implies that

$$E(|z_i^0|^2) = \text{Var}(z_i^0) + |E(z_i^0)|^2 \leq \tau_k^2 + 4C_k(t)^2.$$

For any $g \in S^m(\mathbb{I})$ with $J(g) \leq \rho^2$, we have

$$\begin{aligned} & E\{|\tilde{z} - \tilde{z}^0\}^T A z^0|^2\} \\ &= E\left\{ \left(\sum_{u,v=1}^c a_{u,v} \omega_u z_v^0 \right)^2 \right\} \\ &= \sum_{1 \leq u_1, u_2, v_1, v_2 \leq c} a_{u_1, v_1} a_{u_2, v_2} E\{\omega_{u_1} \omega_{u_2} z_{v_1}^0 z_{v_2}^0\} \\ &= \sum_{\substack{u_1, u_2, v_1 \\ \text{are mutually different}}} a_{u_1, v_1} a_{u_2, v_1} E\{\omega_{u_1}\} E\{\omega_{u_2}\} E\{|z_{v_1}^0|^2\} + \sum_{u_1 \neq v_1} a_{u_1, v_1}^2 E\{\omega_{u_1}^2\} E\{|z_{v_1}^0|^2\} \\ &+ \sum_{u_1 \neq u_2} a_{u_1, u_1} a_{u_2, u_2} E\{\omega_{u_1} z_{u_1}^0\} E\{\omega_{u_2} z_{u_2}^0\} + \sum_{u_1 \neq u_2} a_{u_1, u_1} a_{u_2, u_1} E\{\omega_{u_1} |z_{u_1}^0|^2\} E\{\omega_{u_2}\} \\ &+ \sum_{u_1} a_{u_1, u_1}^2 E\{\omega_{u_1}^2 |z_{u_1}^0|^2\} \equiv T_1 + T_2 + T_3 + T_4 + T_5. \end{aligned}$$

To complete the proof, we will analyze the above T_1 through T_5 terms.

For T_1 , we have

$$\begin{aligned}
 T_1 &= (\tau_k^2 + 4C_k(t)^2) \sum_{u_1 \neq u_2} \sum_{v_1 \neq u_1, u_2} a_{u_1, v_1} a_{u_2, v_1} E\{\omega_{u_1}\} E\{\omega_{u_2}\} \\
 &\leq (\tau_k^2 + 4C_k(t)^2) \sum_{u_1, u_2=1}^n \left(\sum_{v_1 \neq u_1, u_2} a_{u_1, v_1} a_{u_2, v_1} \right) E\{\omega_{u_1}\} E\{\omega_{u_2}\} \\
 &= (\tau_k^2 + 4C_k(t)^2) E\{x\}^T (A - A_0)^2 E\{x\} \\
 &\leq 2(\tau_k^2 + 4C_k(t)^2) E\{x\}^T (A^2 + A_0^2) E\{x\},
 \end{aligned}$$

where recall $A_0 = \text{diag}(a_{1,1}, \dots, a_{c,c})$. Since $A \leq I_c$ and $a_{1,1} = \dots = a_{c,c} \asymp 1/(ch) = o(1)$, we have $A^2 + A_0^2 \leq 2I_c$ (as $c \rightarrow \infty$), which, together with (36), further leads to

$$\begin{aligned}
 T_1 &\leq 4(\tau_k^2 + 4C_k(t)^2) E\{x\}^T E\{x\} \\
 &\leq 4(\tau_k^2 + 4C_k(t)^2) \sum_{i=1}^c (|f(i/c)| + 4C_k(t) + \zeta)^2 \\
 &\leq 4(1 + (ch^2)^{-1})(\tau_k^2 + 4C_k(t)^2) \sum_{i=1}^c (|f(i/c)| + 4C_k(t) + \zeta)^2.
 \end{aligned}$$

For T_2 , we have

$$\begin{aligned}
 T_2 &= (\tau_k^2 + 4C_k(t)^2) \sum_{u_1=1}^c \left(\sum_{v_1 \neq u_1} a_{u_1, v_1}^2 \right) E\{\omega_{u_1}^2\} \\
 &\leq (\tau_k^2 + 4C_k(t)^2) \sum_{u_1=1}^c \left(\sum_{v_1=1}^c a_{u_1, v_1}^2 \right) (|f_{u_1}| + 4C_k(t) + \zeta)^2 \\
 &\leq (1 + (ch^2)^{-1})(\tau_k^2 + 4C_k(t)^2) \sum_{i=1}^c (|f(i/c)| + 4C_k(t) + \zeta)^2,
 \end{aligned}$$

where the last inequality follows from

$$\sum_{v=1}^c a_{i,v}^2 = \frac{1}{c} \sum_{r=0}^{c-1} \frac{\lambda_{d,r}^2}{(\lambda + \lambda_{d',r})^4} \lesssim (ch)^{-1} \rightarrow 0.$$

Here the above " \lesssim " is uniformly of $1 \leq i \leq c$.

For T_3 , Cauchy inequality implies that

$$\begin{aligned}
 T_3 &\leq \left(\sum_{i=1}^c a_{i,i} E\{\omega_i z_i^0\} \right)^2 \\
 &\leq c \sum_{i=1}^c a_{i,i}^2 |E\{\omega_i z_i^0\}|^2 \\
 &\leq c(\tau_k^2 + 4C_k(t)^2) \sum_{i=1}^c a_{i,i}^2 (|f(i/c)| + 4C_k(t) + \zeta)^2 \\
 &\leq (1 + (ch^2)^{-1})(\tau_k^2 + 4C_k(t)^2) \sum_{i=1}^c (|f(i/c)| + 4C_k(t) + \zeta)^2,
 \end{aligned}$$

where the last inequality follows from $ca_{1,1}^2 = \dots = ca_{c,c}^2 \asymp (ch^2)^{-1}$, as $c \rightarrow \infty$.

For T_4 , we have

$$\begin{aligned}
 T_4 &= \sum_{i \neq v} a_{i,i} a_{v,i} E\{\omega_i |z_i^0|^2\} E\{\omega_v\} \\
 &\leq \sum_{i=1}^c a_{i,i} E\{|\omega_i| \cdot |z_i^0|^2\} \sum_{v \neq i} |a_{v,i}| E\{|\omega_v|\} \\
 &\lesssim \frac{\tau_k^2 + 4C_k(t)^2}{ch} \sum_{i=1}^c (|f(i/c)| + 4C_k(t) + \zeta) \sqrt{\sum_{v=1}^c a_{v,i}^2} \sqrt{\sum_{i=1}^c (|f(i/c)| + 4C_k(t) + \zeta)^2} \\
 &\lesssim \frac{\tau_k^2 + 4C_k(t)^2}{(ch)^{3/2}} \sqrt{c} \sum_{i=1}^c (|f(i/c)| + 4C_k(t) + \zeta)^2 \\
 &\lesssim \frac{\tau_k^2 + 4C_k(t)^2}{ch^{3/2}} \sum_{i=1}^c (|f(i/c)| + 4C_k(t) + \zeta)^2 \\
 &\lesssim \frac{\tau_k^2 + 4C_k(t)^2}{ch^2} \sum_{i=1}^c (|f(i/c)| + 4C_k(t) + \zeta)^2 \\
 &\lesssim (1 + (ch^2)^{-1})(\tau_k^2 + 4C_k(t)^2) \sum_{i=1}^c (|f(i/c)| + 4C_k(t) + \zeta)^2.
 \end{aligned}$$

For T_5 , it holds that

$$\begin{aligned}
 T_5 &= \sum_{i=1}^c a_{i,i}^2 E\{\omega_i^2 |z_i^0|^2\} \\
 &\lesssim \frac{\tau_k^2 + 4C_k(t)^2}{(ch)^2} \sum_{i=1}^c (|f(i/c)| + 4C_k(t) + \zeta)^2 \\
 &\leq (1 + (ch^2)^{-1})(\tau_k^2 + 4C_k(t)^2) \sum_{i=1}^c (|f(i/c)| + 4C_k(t) + \zeta)^2.
 \end{aligned}$$

From the above analysis of T_1 through T_5 , we get that as $c \rightarrow \infty$, for any $g \in S^m(\mathbb{I})$ with $J(g) \leq \rho^2$, it follows that

$$E\{|\tilde{z} - \tilde{z}^0\}^T A z^0|^2\} \leq 8(1 + (ch^2)^{-1})(\tau_k^2 + 4C_k(t)^2) \sum_{i=1}^c (|f(i/c)| + 4C_k(t) + \zeta)^2.$$

This proves the desired result. ■

For $\nu = 1, 2, \dots, c$, define $\Phi_\nu = (\varphi_\nu(1/c), \varphi_\nu(2/c), \dots, \varphi_\nu(c/c))^T$. Let $\varepsilon = \exp(2\pi\sqrt{-1}/c)$,

$$x_r = \frac{1}{\sqrt{c}}(1, \varepsilon^r, \varepsilon^{2r}, \dots, \varepsilon^{(c-1)r})^T, \quad r = 0, 1, \dots, c-1,$$

and x_r^* be the conjugate transpose of x_r .

Lemma 15 For $0 \leq r \leq c-1$ and $1 \leq v \leq c-1$, one has that

$$\begin{aligned} x_r^* \Phi_{2(pc+v)-1} &= \sqrt{\frac{c}{2}} (\varepsilon^v I(r=v) + \varepsilon^{-v} I(r+v=c)), \\ x_r^* \Phi_{2(pc+v)} &= \sqrt{-\frac{c}{2}} (\varepsilon^v I(r=v) - \varepsilon^{-v} I(r+v=c)); \end{aligned}$$

and

$$\begin{aligned} x_r^* \Phi_{2(pc+c)-1} &= \sqrt{\frac{c}{2}} I(r=0), \\ x_r^* \Phi_{2(pc+c)} &= 0. \end{aligned}$$

Proof: The proof can be accomplished by direct calculations. For instance, the first case holds by following arguments. For $0 \leq r \leq c-1$ and $1 \leq v \leq c-1$,

$$\begin{aligned} x_r^* \Phi_{2(pc+v)-1} &= \frac{1}{\sqrt{2c}} \sum_{i=1}^c \varepsilon^{-r(i-1)} \cos\left(\frac{2\pi vi}{c}\right) \\ &= \frac{1}{\sqrt{2c}} \sum_{i=1}^c \varepsilon^{-r(i-1)} (\varepsilon^{vi} + \varepsilon^{-vi}) \\ &= \frac{1}{\sqrt{2c}} \sum_{i=0}^{c-1} \varepsilon^{-(r-v)i} \varepsilon^v + \frac{1}{\sqrt{2c}} \sum_{i=0}^{c-1} \varepsilon^{-(r+v)i} \varepsilon^{-v} \\ &= \sqrt{\frac{c}{2}} (\varepsilon^v I(r=v) + \varepsilon^{-v} I(r+v=c)). \end{aligned}$$

The proof of other cases is similar. ■

Let $M = (x_0, x_1, \dots, x_{c-1})$ and $M^* \mathbf{f} = (e_0(f), e_1(f), \dots, e_{c-1}(f))^T$, where $\mathbf{f} = (f(1/c), \dots, f(c/c))^T$. Recall M^* is the conjugate transpose of M . Suppose $f \in S^m(\mathbb{I})$ admits Fourier expansion $f = \sum_{\nu=1}^{\infty} f_{\nu} \varphi_{\nu}$.

Lemma 16 There exists a universal constant $\varrho > 0$ s.t. for any $f \in S^m(\mathbb{I})$,

$$\mathbf{f}^T (I_c - A) \mathbf{f} \leq \varrho c (\lambda + c^{-2m}) J(f).$$

Proof: For simplicity, denote $e_r = e_r(f)$. For $1 \leq r \leq c/2$, we have

$$\begin{aligned} \lambda_{d',r}^2 - \lambda_{d,r} &\leq ((2\pi r)^{-2m} + (2\pi(c-r))^{-2m} + 2\bar{d}_m (2\pi c)^{-2m})^2 - (2\pi r)^{-4m} \\ &\leq ((2\pi r)^{-2m} + (1 + 2^{1-2m} \bar{d}_m) (\pi c)^{-2m})^2 - (2\pi r)^{-4m} \\ &= 2(1 + 2^{1-2m} \bar{d}_m) (2\pi r)^{-2m} (\pi c)^{-2m} + (1 + 2^{1-2m} \bar{d}_m)^2 (\pi c)^{-4m}. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
 & 1 - \frac{\lambda_{d,r}}{(\lambda + \lambda_{d',r})^2} \\
 = & \frac{\lambda^2 + 2\lambda\lambda_{d',r} + \lambda_{d',r}^2 - \lambda_{d,r}}{(\lambda + \lambda_{d',r})^2} \\
 \leq & \frac{2\lambda}{\lambda + \lambda_{d',r}} + \frac{\lambda_{d',r}^2 - \lambda_{d,r}}{(\lambda + \lambda_{d',r})^2} \\
 \leq & (2\lambda + 2(1 + 2^{1-2m}\bar{d}'_m)(\pi c)^{-2m} + 2^{2m}(1 + 2^{1-2m}\bar{d}'_m)^2(\pi c)^{-2m})(2\pi r)^{2m} \\
 \leq & \varrho'_m(\lambda + c^{-2m})(2\pi r)^{2m}, \tag{37}
 \end{aligned}$$

where $\varrho'_m = \max\{2, (2(1 + 2^{1-2m}\bar{d}'_m) + 2^{2m}(1 + 2^{1-2m}\bar{d}'_m)^2)\pi^{-2m}\}$, and \bar{d}'_m is defined in (29).

By Lemma 15 and direct calculations, for $1 \leq r \leq c - 1$, we have

$$\begin{aligned}
 e_r &= \sum_{p=0}^{\infty} \sum_{v=1}^c f_{2(pc+v)-1} x_r^* \Phi_{2(pc+v)-1} + \sum_{p=0}^{\infty} \sum_{v=1}^{\infty} f_{2(pc+v)} x_r^* \Phi_{2(pc+v)} \\
 &= \sum_{p=0}^{\infty} \sum_{v=1}^{c-1} f_{2(pc+v)-1} \left(\sqrt{\frac{c}{2}} \varepsilon^v I(r=v) + \sqrt{\frac{c}{2}} \varepsilon^{-v} I(v+r=c) \right) \\
 &\quad + \sum_{p=0}^{\infty} \sum_{v=1}^{c-1} f_{2(pc+v)} \left(\sqrt{-\frac{c}{2}} \varepsilon^v I(r=v) - \sqrt{-\frac{c}{2}} \varepsilon^{-v} I(r+v=c) \right) \\
 &= \varepsilon^r \sqrt{\frac{c}{2}} \sum_{p=0}^{\infty} (f_{2(pc+r)-1} + f_{2(pc+c-r)-1} + \sqrt{-1} f_{2(pc+r)} - \sqrt{-1} f_{2(pc+c-r)}).
 \end{aligned}$$

Therefore, it holds that

$$|e_r|^2 = \frac{c}{2} \left| \sum_{p=0}^{\infty} (f_{2(pc+r)-1} + f_{2(pc+c-r)-1}) \right|^2 + \frac{c}{2} \left| \sum_{p=0}^{\infty} (f_{2(pc+r)} - f_{2(pc+c-r)}) \right|^2.$$

It is easy to see that

$$|e_r|^2 = |e_{c-r}|^2, \quad r = 1, \dots, c-1.$$

For $1 \leq r \leq c/2$, we have

$$\begin{aligned}
 |e_r|^2 &\leq c \sum_{p=0}^{\infty} f_{2(pc+r)-1}^2 (2\pi(pc+r))^{2m} \sum_{p=0}^{\infty} (2\pi(pc+r))^{-2m} \\
 &\quad + c \sum_{p=0}^{\infty} f_{2(pc+c-r)-1}^2 (2\pi(pc+c-r))^{2m} \sum_{p=0}^{\infty} (2\pi(pc+c-r))^{-2m} \\
 &\quad + c \sum_{p=0}^{\infty} f_{2(pc+r)}^2 (2\pi(pc+r))^{2m} \sum_{p=0}^{\infty} (2\pi(pc+r))^{-2m} \\
 &\quad + c \sum_{p=0}^{\infty} f_{2(pc+c-r)}^2 (2\pi(pc+c-r))^{2m} \sum_{p=0}^{\infty} (2\pi(pc+c-r))^{-2m} \\
 &\leq \left(c \sum_{p=0}^{\infty} f_{2(pc+r)-1}^2 (2\pi(pc+r))^{2m} + c \sum_{p=0}^{\infty} f_{2(pc+c-r)-1}^2 (2\pi(pc+c-r))^{2m} \right. \\
 &\quad \left. + c \sum_{p=0}^{\infty} f_{2(pc+r)}^2 (2\pi(pc+r))^{2m} + c \sum_{p=0}^{\infty} f_{2(pc+c-r)}^2 (2\pi(pc+c-r))^{2m} \right) \\
 &\quad \times \frac{2m}{2m-1} (2\pi r)^{-2m}, \tag{38}
 \end{aligned}$$

where (38) follows by an elementary inequality

$$\sum_{p=0}^{\infty} (2\pi(pc+r))^{-2m} \leq \frac{2m}{2m-1} (2\pi r)^{-2m}, \quad 1 \leq r \leq c/2.$$

Meanwhile, a similar analysis leads to

$$\begin{aligned}
 |e_0|^2 &= \frac{c}{2} \left| \sum_{p=0}^{\infty} f_{2(pc+c)-1} \right|^2 \\
 &\leq \frac{c}{2} \sum_{p=0}^{\infty} f_{2(pc+c)-1}^2 (2\pi(pc+c))^{2m} \sum_{p=0}^{\infty} (2\pi(pc+c))^{-2m} \\
 &= \frac{c^{1-2m}}{2} \sum_{p=0}^{\infty} f_{2(pc+c)-1}^2 (2\pi(pc+c))^{2m} \times \sum_{p=1}^{\infty} (2\pi p)^{-2m}. \tag{39}
 \end{aligned}$$

Now it follows from (37), (38) and (39), and elementary facts $\lambda_{d',r} = \lambda_{d',c-r}$ and $\lambda_{d,r} = \lambda_{d,c-r}$, for $1 \leq r \leq c-1$, that

$$\begin{aligned}
 & \mathbf{f}^T(I_c - A)\mathbf{f} \\
 &= \sum_{r=0}^{c-1} \left(1 - \frac{\lambda_{d,r}}{(\lambda + \lambda_{d',r})^2}\right) |e_r|^2 \\
 &= \left(1 - \frac{\lambda_{d,0}}{(\lambda + \lambda_{d',0})^2}\right) |e_0|^2 + \sum_{1 \leq r \leq c/2} \left(1 - \frac{\lambda_{d,r}}{(\lambda + \lambda_{d',r})^2}\right) |e_r|^2 + \sum_{c/2 < r \leq c-1} \left(1 - \frac{\lambda_{d,r}}{(\lambda + \lambda_{d',r})^2}\right) |e_r|^2 \\
 &\leq |e_0|^2 + 2 \sum_{1 \leq r \leq c/2} \left(1 - \frac{\lambda_{d,r}}{(\lambda + \lambda_{d',r})^2}\right) |e_r|^2 \\
 &\leq \frac{c^{1-2m}}{2} \sum_{p=0}^{\infty} f_{2^{(pc+c)-1}}^2 (2\pi(pc+c))^{2m} \times \sum_{p=1}^{\infty} (2\pi p)^{-2m} \\
 &\quad + 2\varrho'_m c(\lambda + c^{-2m}) \frac{2m}{2m-1} \sum_{1 \leq r \leq c/2} \left(\sum_{p=0}^{\infty} f_{2^{(pc+r)-1}}^2 (2\pi(pc+r))^{2m} \right. \\
 &\quad + \sum_{p=0}^{\infty} f_{2^{(pc+c-r)-1}}^2 (2\pi(pc+c-r))^{2m} + \sum_{p=0}^{\infty} f_{2^{(pc+r)}}^2 (2\pi(pc+r))^{2m} \\
 &\quad \left. + \sum_{p=0}^{\infty} f_{2^{(pc+c-r)}}^2 (2\pi(pc+c-r))^{2m} \right) \\
 &\leq \varrho_m c(\lambda + c^{-2m}) \sum_{\nu=1}^{\infty} (f_{2^{\nu-1}}^2 + f_{2^\nu}^2) (2\pi\nu)^{2m} \\
 &= \varrho_m c(\lambda + c^{-2m}) J(f),
 \end{aligned}$$

where $\varrho_m = \max\{\sum_{p=1}^{\infty} (2\pi p)^{-2m}/2, 4m\varrho'_m/(2m-1)\}$. It is straightforward to see ϱ_m is a decreasing function with respect to m , therefore, we choose $\varrho = \varrho_{m=1}$. This proves Lemma 16. \blacksquare

The proof of Theorem 8 requires some recent Gaussian approximation result, i.e., Theorem 3.1 in Koike (2019).

Lemma 17 *For each $c \in \mathbb{N}$, let Ψ_c be an c -dimensional centered Gaussian vector with covariance matrix $\Sigma_c = (\Sigma_c(m, m'))_{1 \leq m, m' \leq c}$ and $m_u \geq 2$ be an integer. Also, for each $m = m_l, \dots, m_u$, let A_m be an $c \times c$ symmetric matrix and $Z_c = (Z_{c, m_l}, \dots, Z_{c, m_u})^\top$ be an $m_u - m_l + 1$ -dimensional centered Gaussian vector with covariance matrix $\mathfrak{C}_c = (\mathfrak{C}_c(m, m'))_{m_l \leq m, m' \leq m_u}$. Set $F_{c,m} := \Psi_c^\top A_m \Psi_c - E[\Psi_c^\top A_m \Psi_c]$ and suppose that the following conditions are satisfied:*

1. *There is a constant $b > 0$ such that $\mathfrak{C}_c(m, m) \geq b$ for every c and every $m = m_l, \dots, m_u$.*

2. $\max_{m_l \leq m \leq m_u} \left(E [F_{c,m}^4] - 3E [F_{c,m}^2]^2 \right) \log^6 m_c \rightarrow 0$ as $c \rightarrow \infty$.
3. $\max_{m_l \leq m, m' \leq m_u} |\mathfrak{C}_c(m, m') - E [F_{c,m} F_{c,m'}]| \log^2 m_c \rightarrow 0$ as $c \rightarrow \infty$.

Then we have

$$\sup_{x \in \mathbb{R}} \left| P \left(\max_{m_l \leq m \leq m_u} F_{c,m} \leq x \right) - P \left(\max_{m_l \leq m \leq m_u} Z_{c,m} \leq x \right) \right| \rightarrow 0, \text{ as } c \rightarrow \infty.$$

Proof This is Theorem 3.1 in Koike (2019). ■

The proof of Theorem 9 requires some rate conditions which are summarized in the following lemma.

Lemma 18 *Suppose $\lambda_m = a_n^{2m} n^{-4m/(4m+1)} \log(m_u)^{2m/(4m+1)}$, then for any $m_l < m < m_u \rightarrow \infty$, under Condition (C), the following rate conditions hold:*

$$h_m \log^2(m_u) \rightarrow 0; m_u \log^2(m_u)/ch_m \rightarrow 0;$$

$$(h_{m'}/h_m)^{-1/2} m_u^2 \log^2(m_u) \rightarrow 0, \text{ for any } m_l \leq m < m' < m_u;$$

$$ch_m / \log(c) \rightarrow \infty; h_m^{1/2} \log(c) \rightarrow 0,$$

where $h_m = \lambda_m^{-1} = a_n n^{-2/(4m+1)} \log(m_u)^{1/(4m+1)}$.

Proof It is easy to see $h_{m_l} < h_m < h_{m_u}$. Therefore

$$h_m \log^2(m_u) < h_{m_u} \log^2(m_u) \lesssim a_n n^{-2/(4m_u+1)} [\log(n)]^2 \rightarrow 0.$$

$$m_u \log^2(m_u)/ch_m < m_u \log^2(m_u)/ch_{m_l} \lesssim n^{2/(4m_l+1)} \log(n)/(ca_n) \rightarrow 0.$$

$$\begin{aligned} (h_{m'}/h_m)^{-1/2} m_u^2 \log^2(m_u) &= n^{-\frac{4(m'-m)}{(4m+1)(4m'+1)}} (\log(m_u))^{-\frac{2m'-2m}{(4m+1)(4m'+1)}} m_u^2 \log^2(m_u) \\ &\lesssim n^{-\frac{4}{(4m_u+1)^2}} (\log(m_u))^{-\frac{2}{(4m_u+1)^2}} m_u^2 \log^2(m_u) \rightarrow 0. \end{aligned}$$

where the last “ $\rightarrow 0$ ” follows from the assumption $m_u \lesssim \log^{d_0}(n)$ for some $d_0 \in (0, 1/2)$. For the last two terms, one has that

$$ch_m / \log(c) > ch_{m_l} / \log(c) \gtrsim \frac{ca_n}{n^{2/(4m_l+1)} \log(n)} \rightarrow \infty.$$

$$h_m^{1/2} \log(c) < h_{m_u}^{1/2} \log(c) \lesssim a_n^{1/2} n^{-1/(4m_u+1)} \log(n) \rightarrow 0.$$

■

Appendix D. Proofs for main theorems

Proof of Theorem 1:

It holds that $\|\widehat{g}_{\mu,t,c}^B - g_0\|^2 \leq 2\|\widehat{g}_{\mu,t,c}^B - \widehat{g}_c^{\text{ss}}\|^2 + 2\|\widehat{g}_c^{\text{ss}} - g_0\|^2$, and we analyze these two terms separately. We first analyze $\|\widehat{g}_{\mu,t,c}^B - \widehat{g}_c^{\text{ss}}\|^2$. Because

$$|Q(y_j) - y_j| \leq C_k(t) \mathbb{1} \left\{ y_j \in \cup_{j=2}^{k-1} R_j(t) \right\} + |y_j - \mu_1| \mathbb{1} \{y_j \in R_1(t)\} + |y_j - \mu_k| \mathbb{1} \{y_j \in R_k(t)\},$$

we have

$$\begin{aligned} (z_i - \widetilde{y}_i)^2 &= \left\{ z_i - \frac{1}{\widetilde{n}} \sum_{j=(i-1)\widetilde{n}+1}^{\widetilde{i}\widetilde{n}} Q(y_j) + \frac{1}{\widetilde{n}} \sum_{j=(i-1)\widetilde{n}+1}^{\widetilde{i}\widetilde{n}} Q(y_j) - \widetilde{y}_i \right\}^2 \\ &\leq 2 \left\{ \frac{1}{\widetilde{n}} \sum_{j=(i-1)\widetilde{n}+1}^{\widetilde{i}\widetilde{n}} (Q(y_j) - y_j) \right\}^2 + 2C_k(t)^2 \\ &\leq 2 \left\{ \frac{1}{\widetilde{n}} \sum_{j=(i-1)\widetilde{n}+1}^{\widetilde{i}\widetilde{n}} C_k(t) \mathbb{1} \left\{ y_j \in \cup_{j=2}^{k-1} R_j(t) \right\} \right. \\ &\quad \left. + |y_j - \mu_1| \mathbb{1} \{y_j \in R_1(t)\} + |y_j - \mu_k| \mathbb{1} \{y_j \in R_k(t)\} \right\}^2 + 2C_k(t)^2 \\ &\leq 4C_k(t)^2 + \frac{2}{\widetilde{n}} \sum_{j=(i-1)\widetilde{n}+1}^{\widetilde{i}\widetilde{n}} (y_j - \mu_1)^2 \mathbb{1} \{y_j \in R_1(t)\} + \frac{2}{\widetilde{n}} \sum_{j=(i-1)\widetilde{n}+1}^{\widetilde{i}\widetilde{n}} (y_j - \mu_k)^2 \mathbb{1} \{y_j \in R_k(t)\}. \end{aligned}$$

Therefore, from Lemma 10, we have

$$\begin{aligned} E\|\widehat{g}_{\mu,t,c}^B - \widehat{g}_c^{\text{ss}}\|^2 &\leq c^{-1} \sum_{i=1}^c E\{(z_i - \widetilde{y}_i)^2\} \\ &\leq 4C_k(t)^2 + \frac{2}{n} \sum_{i=1}^n E\{(y_i - \mu_1)^2 \mathbb{1} \{y_i \in R_1(t)\}\} + \frac{2}{n} \sum_{i=1}^n E\{(y_i - \mu_k)^2 \mathbb{1} \{y_i \in R_k(t)\}\}. \end{aligned}$$

On the other hand, by elementary calculations we have

$$\begin{aligned} \frac{2}{n} \sum_{i=1}^n E\{(y_i - \mu_1)^2 \mathbb{1} \{y_i \in R_1(t)\}\} &= \frac{2}{n} \sum_{i=1}^n \int_{-\infty}^{t_1} (z - \mu_1)^2 p\left(\frac{z - g_0(i/n)}{\sigma}\right) \sigma^{-1} dz, \\ \frac{2}{n} \sum_{i=1}^n E\{(y_i - \mu_k)^2 \mathbb{1} \{y_i \in R_k(t)\}\} &= \frac{2}{n} \sum_{i=1}^n \int_{t_{k-1}}^{\infty} (z - \mu_k)^2 p\left(\frac{z - g_0(i/n)}{\sigma}\right) \sigma^{-1} dz, \end{aligned}$$

where $p(\cdot)$ is the distribution of ϵ . Combining the above, we get

$$E\|\widehat{g}_{\mu,t,c}^B - \widehat{g}_c^{\text{ss}}\|^2 \leq G_{c,k}(t). \quad (40)$$

Next, we analyze the mean square error of the second term $\|\widehat{g}_c^{\text{ss}} - g_0\|^2$. For the sake of theoretical investigation, we introduce the following function,

$$g_{\text{new}} = \sum_{i=1}^c \theta_{\text{new},i} K_{i/c},$$

where $(\theta_{\text{new},1}, \dots, \theta_{\text{new},c})^T = c^{-1}(\Sigma_c + \lambda I_c)^{-1} \tilde{f}$ with $\tilde{f} = (f(1/c), \dots, f(c/c))^T \in \mathbb{R}^c$, and $f(x)$ is the integral function of g_0 as defined in (15), i.e.,

$$f(x) = \frac{1}{2\Delta} \int_{\max(x-\Delta, 0)}^{\min(x+\Delta, 1)} g_0(s) ds, \text{ where } x \in [0, 1], \text{ and } \Delta = \frac{1}{c}.$$

Recall that $\hat{g}_c^{\text{ss}} = \sum_{i=1}^c \tilde{\theta}_i K_{i/c} = c^{-1} \sum_{\nu=1}^{\infty} \frac{\Phi_\nu^T(\Sigma_c + \lambda I_c)^{-1} \tilde{y}}{\gamma_\nu} \varphi_\nu$, where $(\tilde{\theta}_1, \dots, \tilde{\theta}_c)^T = c^{-1}(\Sigma_c + \lambda I_c)^{-1} \tilde{y}$ with $\tilde{y} = (\tilde{y}_1, \dots, \tilde{y}_c)^T$, $\varphi_{2k-1}(x) = \sqrt{2} \cos(2\pi kx)$, $\varphi_{2k}(x) = \sqrt{2} \sin(2\pi kx)$ are the trigonometric basis functions, $\gamma_{2k-1} = \gamma_{2k} = (2\pi k)^{2m}$, and $\Phi_\nu = (\varphi_\nu(1/c), \varphi_\nu(2/c), \dots, \varphi_\nu(c/c))^T$. Therefore, we have

$$\begin{aligned} \|E(\hat{g}_c^{\text{ss}}) - g_{\text{new}}\|^2 &= c^{-1}(\tilde{g} - \tilde{f})^T (\Sigma_c + \lambda I_c)^{-1} \Omega_c (\Sigma_c + \lambda I_c)^{-1} (\tilde{g} - \tilde{f}) \\ &\leq c^{-1}(\tilde{g} - \tilde{f})^T (\tilde{g} - \tilde{f}) \\ &= O(n^{-2}), \end{aligned} \tag{41}$$

where $\tilde{g} = (\tilde{g}_1, \dots, \tilde{g}_c)^T$ and $\tilde{g}_i = \frac{\sum_{j=(i-1)\tilde{n}+1}^{i\tilde{n}} g_0(j/n)}{\tilde{n}}$, $i = 1, \dots, c$. Next, we evaluate $E(\|\hat{g}_c^{\text{ss}} - E(\hat{g}_c^{\text{ss}})\|^2)$. Note that Σ_c, Ω_c can be decomposed as $\Sigma_c = M \Lambda_{d'} M^*$, $\Omega_c = M \Lambda_d M^*$, as defined in (26). Furthermore, we let $\tilde{y}^0 = (\tilde{y}_1^0, \dots, \tilde{y}_c^0)^T$, where $\tilde{y}_i^0 = \frac{1}{\tilde{n}} \sum_{j=(i-1)\tilde{n}+1}^{i\tilde{n}} \sigma \epsilon_j$. Hence, we obtain

$$\begin{aligned} E(\|\hat{g}_c^{\text{ss}} - E(\hat{g}_c^{\text{ss}})\|^2) &= \frac{1}{c^2 \gamma_\nu^2} \sum_{\nu=1}^{\infty} E \left\{ \left| \Phi_\nu^T (\Sigma_c + \lambda I_c)^{-1} \tilde{y}^0 \right|^2 \right\} \\ &= \frac{\sigma^2}{c^2 \gamma_\nu^2} \sum_{\nu=1}^{\infty} \text{trace} \left((\Sigma_c + \lambda I_c)^{-1} \Phi_\nu \Phi_\nu^T (\Sigma_c + \lambda I_c)^{-1} \right) E \tilde{\epsilon}^2 \\ &= \frac{\sigma^2}{n} \text{trace} \left((\Sigma_c + \lambda I_c)^{-1} \sum_{\nu=1}^{\infty} \frac{\Phi_\nu \Phi_\nu^T / c}{\gamma_\nu^2} (\Sigma_c + \lambda I_c)^{-1} \right) \\ &= \frac{\sigma^2}{n} \text{trace} \left((\Sigma_c + \lambda I_c)^{-1} \Omega_c (\Sigma_c + \lambda I_c)^{-1} \right) \\ &= \frac{\sigma^2}{n} \text{trace} \left(M (\Lambda_{d'} + \lambda I_c)^{-1} \Lambda_d (\Lambda_{d'} + \lambda I_c)^{-1} M^* \right) \\ &= \frac{\sigma^2}{n} \sum_{l=0}^{c-1} \frac{\lambda_{d,l}}{(\lambda + \lambda_{d',l})^2}. \end{aligned}$$

By expressions of $\lambda_{d,l}$'s, the above is upper bounded by the following

$$\begin{aligned}
 & \frac{2\sigma^2\bar{d}_m}{n(2\bar{d}'_m + (2\pi c)^{2m}\lambda)^2} + \sigma^2(1 + \bar{d}_m)n^{-1} \sum_{l=1}^{c-1} \frac{(2\pi(c-l))^{-4m} + (2\pi l)^{-4m}}{(\lambda + (2\pi(c-l))^{-2m} + (2\pi l)^{-2m})^2} \\
 & \leq \frac{2\sigma^2\bar{d}_m}{n(2\bar{d}'_m + (2\pi c)^{2m}\lambda)^2} + 2\sigma^2(1 + \bar{d}_m)n^{-1} \sum_{1 \leq l \leq n/2} \frac{(2\pi l)^{-4m} + (2\pi(c-l))^{-4m}}{(\lambda + (2\pi l)^{-2m} + (2\pi(c-l))^{-2m})^2} \\
 & \leq \frac{2\sigma^2\bar{d}_m}{n(2\bar{d}'_m + (2\pi c)^{2m}\lambda)^2} + 4\sigma^2(1 + \bar{d}_m)n^{-1} \sum_{1 \leq l \leq n/2} \frac{(2\pi l)^{-4m}}{(\lambda + (2\pi l)^{-2m})^2} \\
 & \leq \frac{2\sigma^2\bar{d}_m}{n(2\bar{d}'_m + (2\pi c)^{2m}\lambda)^2} + \frac{2\sigma^2(1 + \bar{d}_m)}{\pi nh} \int_0^{\pi ch} \frac{1}{(1+x^{2m})^2} dx \\
 & \leq b_m \left(\frac{\sigma^2}{n} + \frac{\sigma^2}{nh} \int_0^{\pi ch} \frac{1}{(1+x^{2m})^2} dx \right),
 \end{aligned}$$

where $b_m \geq 1$ is a constant only depending on m . From the above analysis, we obtain

$$E(\|\widehat{g}_c^{\text{ss}} - E(\widehat{g}_c^{\text{ss}})\|^2) \lesssim n^{-1} + (nh)^{-1}.$$

Using above analysis and (41), we have

$$E(\|\widehat{g}_c^{\text{ss}} - g_{\text{new}}\|^2) \lesssim n^{-1} + (nh)^{-1}. \quad (42)$$

Now, we consider the difference between original regression function g_0 and the integral function f defined in (15), i.e., $\|f - g_0\|^2$. By definition, for $t \in [\Delta, 1 - \Delta]$, there exists t' between $t - \Delta$ and $t + \Delta$ such that

$$\begin{aligned}
 f(t) - g_0(t) &= \frac{1}{2\Delta} \int_{t-\Delta}^{t+\Delta} (g_0(s) - g_0(t)) ds \\
 &= \frac{1}{2\Delta} \int_{t-\Delta}^{t+\Delta} \left(g_0'(t)(s-t) + \frac{g_0''(t')}{2}(s-t)^2 \right) ds \\
 &= \frac{1}{2\Delta} \int_{t-\Delta}^{t+\Delta} \frac{g_0''(t')}{2}(s-t)^2 ds = \frac{\Delta^2}{3} g_0''(t').
 \end{aligned}$$

On the other hand, for $t \in [0, \Delta]$, there exists t' between 0 and $t + \Delta$ such that

$$\begin{aligned}
 f(t) - g_0(t) &= \frac{1}{2\Delta} \int_0^{t+\Delta} (g_0(s) - g_0(t)) ds \\
 &= \frac{1}{2\Delta} \int_0^{t+\Delta} (g_0'(t')(s-t)) ds \leq \frac{\Delta}{4} g_0'(t').
 \end{aligned}$$

In a similar way, we obtain $f(t) - g_0(t) \leq \frac{\Delta}{4} g_0'(t')$ for $t \in [t - \Delta, 1]$ and some $t' \in [t - \Delta, 1]$. Therefore, by Sobolev inequality, we know $\int_0^1 |g_0''(t)|^2 dt \leq \int_0^1 |g_0^{(m)}(t)|^2 dt < \infty$ and

$\int_0^1 |g'_0(t)|^2 dt \leq \int_0^1 |g_0^{(m)}(t)|^2 dt < \infty$, which implies

$$\begin{aligned} \|f - g_0\|^2 &= \int_0^1 |f(t) - g(t)|^2 dt \\ &= \int_{t-\Delta}^{t+\Delta} |f(t) - g(t)|^2 dt + \int_0^{t+\Delta} |f(t) - g(t)|^2 dt + \int_{t-\Delta}^1 |f(t) - g(t)|^2 dt \\ &= O(c^{-3}). \end{aligned} \quad (43)$$

In the end, because both g_{new} and f belong to Sobolev space, and g_{new} can be viewed as the approximate error of spline estimates with respect to f without random error. By classical spline theory ((Wahba, 1990)), we know

$$\|g_{\text{new}} - f\|^2 = O(c^{-2m} + \lambda). \quad (44)$$

As a consequence, from (42), (43), and (44), we have $E\|\widehat{g}_c^{\text{ss}} - g_0\|^2 \leq 3E(\|\widehat{g}_c^{\text{ss}} - g_{\text{new}}\|^2) + 3\|g_{\text{new}} - f\|^2 + 3\|f - g_0\|^2 = O((nh)^{-1} + c^{-3} + c^{-2m} + \lambda)$. Combining the result in (40), we get the desired result. \blacksquare

Proof of Corollary 2: Because as $|t_1| \rightarrow \infty$,

$$\begin{aligned} G_{c,k,1}(t) &= \frac{1}{n} \sum_{i=1}^n \int_{-\infty}^{t_1} (z - \mu_1)^2 \sigma^{-1} p\left(\frac{z - g_0(i/n)}{\sigma}\right) dz \\ &\lesssim 2n^{-1} \sigma^{-1} \int_{-\infty}^{\frac{t_1 - g_0(i/n)}{\sigma}} z^2 p(z) dz + 2n^{-1} \sigma^{-1} \int_{-\infty}^{\frac{t_1 - g_0(i/n)}{\sigma}} \mu_1^2 p(z) dz. \end{aligned}$$

The first term in the above equation is bounded by $n^{-2m/(2m+1)}$ because $p(z)$ satisfies $\int_{|z| \geq T} z^2 p(z) dz = O(\exp(-T^d))$ and $d \geq \frac{4m}{2m+1}$, $|t_1| \asymp \sqrt{\log(n)}$. Due to Condition **(B)**, we know $G_{c,k,1}(t) = O(n^{-2m/(2m+1)})$. Similarly, we know $G_{c,k,2}(t) = O(n^{-2m/(2m+1)})$. Hence $G_{c,k}(t) = C_k(t)^2 + G_{c,k,1}(t) + G_{c,k,2}(t) = O(n^{-2m/(2m+1)})$. The result follows by Theorem 1 and $\lambda \asymp n^{-2m/(2m+1)}$, $c \asymp n^{\frac{\max\{1, 2m/3\}}{2m+1}}$. \blacksquare

Proof of Theorem 3: Suppose z_i^* 's are the quantized samples corresponding to $\mu_j = \mu_j^*$ for $1 \leq j \leq k$, where μ_j^* are defined by

$$\mu_j^* = \frac{\sum_{i=1}^n E\{y_i I(y_i \in R_j(t))\}}{\sum_{i=1}^n P(y_i \in R_j(t))}. \quad (45)$$

For $p > 0$, define the p th order moment of the standardized z_i^* :

$$m_p = E_{H_0}\{|z_i^* / \tau_k^*|^p\}, \quad (46)$$

where E_{H_0} denotes the expectation under H_0 and $\tau_k^{*2} = \text{Var}(z_i^* | H_0)$. Because $|\mu_j - \mu_j^*| \leq C_k(t)$ for $j = 2, \dots, k-1$, and under Condition **(B)**, we have that $\tau_k^{*2} = O(cn^{-1})$. Furthermore, since $b \gg \log_2(\sqrt{n\mathcal{T}_n h^{1/2}})$, which implies that $C_k(t)^4 \asymp \frac{\mathcal{T}_n^2}{2^{4b}} \ll \frac{\mathcal{T}_n^2}{n^2 h \mathcal{T}_n^2} =$

$(n^2h)^{-1} = o(c^2n^{-2})$ and the assumption that $E([\tilde{n}^{-1} \sum_{j=1}^{\tilde{n}} Q(\epsilon_j)]^4) = O(c^2n^{-2})$, one has that $m_p = O(1)$ for $p = 3, 4$.

Define $z_i^{\text{sd}} = z_i^*/\tau_k^*$ for $i = 1, \dots, c$. Then z_i^{sd} are *iid* variables with zero-mean and unit variance. Define $z^* = (z_1^*, \dots, z_c^*)^T$ and $z^{\text{sd}} = (z_1^{\text{sd}}, \dots, z_c^{\text{sd}})^T$. Define $A_0 = \text{diag}(a_{1,1}, \dots, a_{c,c})$ and $A_1 = A - A_0$. Let $B = A_1/s_c$. Define $\alpha_l = \frac{\lambda_{d,l}}{(\lambda + \lambda_{d',l})^2}$, $l = 0, \dots, c-1$. Immediately, for all $i = 1, \dots, c$, $a_{i,i} = c^{-1} \sum_{l=0}^{c-1} \alpha_l \asymp 1/(ch)$, therefore,

$$\begin{aligned} \sum_{i \neq i'} a_{i,i'}^2 &= \sum_{i,i'=1}^c a_{i,i'}^2 - \sum_{i=1}^c a_{i,i}^2 \\ &= \text{trace}(A^2) - c \left(c^{-1} \sum_{l=0}^{c-1} \alpha_l \right)^2 \\ &= \sum_{l=0}^{c-1} \alpha_l^2 - c \left(c^{-1} \sum_{l=0}^{c-1} \alpha_l \right)^2 \\ &\asymp h^{-1}(1 - 1/(ch)) \asymp h^{-1}, \end{aligned}$$

where the last " \asymp " follows from condition $(ch)^{-1} = o(1)$. This implies that $s_c^2 \asymp h^{-1}$. Furthermore,

$$\text{trace}(A^2) = \sum_{l=0}^{c-1} \alpha_l^2 \asymp h^{-1} \quad \text{and} \quad \text{trace}(A^4) = \sum_{l=0}^{c-1} \alpha_l^4 \asymp h^{-1}. \quad (47)$$

Let $T_{\mu^*,t,c}^*$ be the test statistic corresponding to z_i^* 's. By (25) it can be shown that $cT_{\mu^*,t,c}^* = z^{*T} A z^*$, which leads to that

$$\begin{aligned} \frac{cT_{\mu^*,t,c}^* - \sum_{i=1}^c a_{i,i} \tau_k^{*2}}{s_c \tau_k^{*2}} &= \frac{z^{*T} A z^* - \sum_{i=1}^c a_{i,i} \tau_k^{*2}}{s_c \tau_k^{*2}} \\ &= \frac{(z^{\text{sd}})^T A z^{\text{sd}} - \sum_{i=1}^c a_{i,i}}{s_c} \\ &= \frac{\sum_{i=1}^c a_{i,i} ((z_i^{\text{sd}})^2 - 1) + \sum_{1 \leq i \neq i' \leq c} a_{i,i'} z_i^{\text{sd}} z_{i'}^{\text{sd}}}{s_c} \\ &= \frac{\sum_{i=1}^c a_{i,i} ((z_i^{\text{sd}})^2 - 1)}{s_c} + \sum_{i \neq i'} b_{i,i'} z_i^{\text{sd}} z_{i'}^{\text{sd}} \equiv Q_1 + Q_2. \end{aligned}$$

We first look at Q_1 . By (46) we have

$$E\left\{ \left| \sum_{i=1}^c a_{i,i} ((z_i^{\text{sd}})^2 - 1) \right|^2 / s_c^2 \right\} = s_c^{-2} \sum_{i=1}^c a_{i,i}^2 (m_4 - 1) \asymp \frac{m_4 - 1}{ch} = o(1),$$

which leads to $Q_1 = o_P(1)$.

Define $b_{i,i} = 0$ for $i = 1, \dots, c$ and $B = [b_{i,i'}]_{1 \leq i, i' \leq c}$. We next analyze Q_2 . Note that $Q_2 = (z^{\text{sd}})^T B z^{\text{sd}}$. Let $(\tilde{z}_1^{\text{sd}}, \dots, \tilde{z}_c^{\text{sd}})^T$ be an independent copy of $z^{\text{sd}} = (z_1^{\text{sd}}, \dots, z_c^{\text{sd}})^T$.

Let I be uniform distributed on $\{1, 2, \dots, c\}$. Throughout, we let \tilde{z}_i^{sd} , z_i^{sd} and I be mutually independent. Define $\tilde{z}^{\text{sd}} = (z_1^{\text{sd}}, \dots, z_{I-1}^{\text{sd}}, \tilde{z}_I^{\text{sd}}, z_{I+1}^{\text{sd}}, \dots, z_c^{\text{sd}})^T$. So $(z^{\text{sd}}, \tilde{z}^{\text{sd}})$ is an exchangeable pair (see Reinert and Röllin (2009)), and $\tilde{z}^{\text{sd}} = z^{\text{sd}} + e_I(\tilde{z}_I^{\text{sd}} - z_I^{\text{sd}})$, where $e_j = (0, \dots, 0, 1, 0, \dots, 0)^T$ with 1 being at the j th position for $j = 1, \dots, c$. Let $Q'_2 = ((\tilde{z}^{\text{sd}})^T B \tilde{z}^{\text{sd}})$. By a simple calculation it can be shown that $Q'_2 - Q_2 = (\tilde{z}^{\text{sd}})^T B \tilde{z}^{\text{sd}} - (z^{\text{sd}})^T B z^{\text{sd}} = 2(\tilde{z}_I^{\text{sd}} - z_I^{\text{sd}})e_I^T B z^{\text{sd}}$. So it follows that

$$E\{Q'_2 - Q_2 | z^{\text{sd}}\} = E\{2(\tilde{z}_I^{\text{sd}} - z_I^{\text{sd}})e_I^T B z^{\text{sd}} | z^{\text{sd}}\} = \frac{2}{c} \sum_{j=1}^c E\{(\tilde{z}_j^{\text{sd}} - z_j^{\text{sd}})e_j^T B z^{\text{sd}} | z^{\text{sd}}\} = -\frac{2}{c} Q_2. \quad (48)$$

Let $g_0^* : \mathbb{R} \rightarrow [0, 1]$ be a C^3 -function such that $g_0^*(s) = 1$ for $s \leq 0$ and $g_0^*(s) = 0$ for $s \geq 1$. Let $G_u(s) = g_0^*(\psi_c(s - u))$ for $u \in \mathbb{R}$, where ψ_c is a positive sequence tending to infinity and satisfying

$$(m_4^2 + m_3^2 + m_4)\psi_c^2 h = o(1), \quad m_3 \psi_c^3 h^{1/2} = o(1). \quad (49)$$

The existence of such ψ_c follows by (46).

Next we will approximate $E\{G_u(Q_2) - G_u(V)\}$ where $V \sim N(0, 1)$. Consider Stein's equation

$$G_u(s) - E\{G_u(V)\} = \ddot{g}(s) - z^{\text{sd}} \dot{g}(s), \quad (50)$$

where \dot{g} and \ddot{g} represent first- and second-order derivatives of g . By Goldstein and Rinott (1996), a solution to (50) is

$$g(s) = - \int_0^1 \frac{1}{2t} [E\{G_u(\sqrt{ts} + \sqrt{1-t}V)\} - E\{G_u(V)\}] dt. \quad (51)$$

Let $C_1 = \|\dot{g}_0^*\|_{\text{sup}}$, $C_2 = \|\ddot{g}_0^*\|_{\text{sup}}$, and $C_3 = \|\dddot{g}_0^*\|_{\text{sup}}$, where \dddot{g}_0^* is the third-order derivative of g_0^* . It is easy to see that

$$\ddot{g}(s) = -\frac{1}{2} \int_0^1 E_U \{\ddot{G}_u(\sqrt{ts} + \sqrt{1-t}U)\} dt,$$

$$\dddot{g}(s) = -\frac{1}{2} \int_0^1 \sqrt{t} E_U \{\dddot{G}_u(\sqrt{ts} + \sqrt{1-t}U)\} dt.$$

Clearly, it holds that $\|\ddot{g}\|_{\text{sup}} \leq \|\ddot{G}_u\|_{\text{sup}} \leq C_2 \psi_c^2$ and $\|\dddot{g}\|_{\text{sup}} \leq \|\dddot{G}_u\|_{\text{sup}} \leq C_3 \psi_c^3$.

By exchangeability, $\frac{1}{2} E\{(Q'_2 - Q_2)(\dot{g}(Q'_2) + \dot{g}(Q_2))\} = 0$. So $E\{(Q'_2 - Q_2)\dot{g}(Q_2)\} + \frac{1}{2} E\{(Q'_2 - Q_2)(\dot{g}(Q'_2) - \dot{g}(Q_2))\} = 0$. Since $E\{(Q'_2 - Q_2)\dot{f}(Q_2)\} = E\{E\{Q'_2 - Q_2 | w\}\dot{g}(Q_2)\} =$

$-\frac{2}{c}E\{Q_2\dot{g}(Q_2)\}$, we have

$$\begin{aligned}
 & E\{Q_2\dot{g}(Q_2)\} - E\{\dot{g}(Q_2)\} \\
 = & \frac{c}{4}E\{(Q'_2 - Q_2)(\dot{g}(Q'_2)) - \dot{g}(Q_2)\} - E\{\dot{g}(Q_2)\} \\
 = & \frac{c}{4}E\{\ddot{g}(Q_2)(Q'_2 - Q_2)^2\} - E\{\dot{g}(Q_2)\} + \frac{c}{4}\int_0^1 (1-t) \times E\{\ddot{g}(Q_2 + t(Q'_2 - Q_2))(Q'_2 - Q_2)^3\}dt \\
 = & E\{\dot{g}(Q_2)(\sum_{i=1}^c (\tilde{z}_i^{\text{sd}} - z_i^{\text{sd}})^2 (e_i^T B z^{\text{sd}})^2 - 1)\} \\
 & + 2\int_0^1 (1-t)E\{\ddot{g}(Q_2 + t(Q'_2 - Q_2))\sum_{i=1}^c (\tilde{z}_i^{\text{sd}} - z_i^{\text{sd}})^3 (e_i^T B z^{\text{sd}})^3\} \\
 \equiv & J_1 + J_2. \tag{52}
 \end{aligned}$$

Next, we analyze J_1 and J_2 separately. Let $M_p = E\{(z_i^{\text{sd}})^p\}$ for $p \geq 1$. For J_1 , by direct examinations we have

$$|J_1| \leq C_2 \psi_c^2 E\left\{\left|\sum_{i=1}^c (1 + (z_i^{\text{sd}})^2)(e_i^T B z^{\text{sd}})^2 - 1\right|\right\} \leq C_2 \psi_c^2 E\left\{\left|\sum_{i=1}^c D_i - 1\right|^2\right\}^{1/2},$$

where $D_i = (1 + (z_i^{\text{sd}})^2)(e_i^T B z^{\text{sd}})^2$. Since $\sum_{i=1}^c E\{D_i\} = 1$, we get that

$$\begin{aligned}
 E\left\{\left(\sum_{i=1}^c D_i - 1\right)^2\right\} &= E\left\{\left|\sum_{i=1}^c [D_i - E\{D_i}]\right|^2\right\} \\
 &= \sum_{i=1}^c E\{(D_i - E\{D_i})^2\} + \sum_{i \neq i'} E\{(D_i - E\{D_i})(D_{i'} - E\{D_{i'}})\}. \tag{53}
 \end{aligned}$$

The first term of (53) is equal to

$$\begin{aligned}
 \sum_{i=1}^c [3(3 + M_4)(e_i^T B^2 e_i)^2 - 4(e_i^T B^2 e_i)^2] &= (5 + 3M_4) \sum_{i=1}^c (e_i^T B^2 e_i)^2 \\
 &= (5 + 3M_4) \sum_{i=1}^c \left(\sum_{l=1}^c b_{i,l}^2\right)^2 \\
 &\leq (5 + 3M_4) \sum_{i=1}^c \left(\sum_{l=1}^c a_{i,l}^2\right)^2 / s_c^4 \\
 &\leq (5 + 3M_4) \text{trace}(A^4) / s_c^4 \asymp (5 + 3M_4)h,
 \end{aligned}$$

where the last “ \asymp ” follows by (47).

The second term of (53) is equal to

$$\begin{aligned}
 & \sum_{i \neq i'} \left(E\{(1 + (z_i^{\text{sd}})^2)(1 + (z_{i'}^{\text{sd}})^2)(e_i^T B z^{\text{sd}})^2 (e_{i'}^T B z^{\text{sd}})^2\} \right. \\
 & \quad \left. - E\{(1 + (z_i^{\text{sd}})^2)(e_i^T B z^{\text{sd}})^2\} E\{(1 + (z_{i'}^{\text{sd}})^2)(e_{i'}^T B z^{\text{sd}})^2\} \right) \\
 = & \sum_{i \neq i'} \left(E\{(1 + (z_i^{\text{sd}})^2)(1 + (z_{i'}^{\text{sd}})^2) E\{(e_i^T B z^{\text{sd}})^2 (e_{i'}^T B z^{\text{sd}})^2 | z_i^{\text{sd}}, z_{i'}^{\text{sd}}\}\} \right. \\
 & \quad \left. - E\{(1 + (z_i^{\text{sd}})^2)(e_i^T B z^{\text{sd}})^2\} E\{(1 + (z_{i'}^{\text{sd}})^2)(e_{i'}^T B z^{\text{sd}})^2\} \right).
 \end{aligned}$$

We have that

$$\begin{aligned}
 E\{(e_i^T B z^{\text{sd}})^2 (e_{i'}^T B z^{\text{sd}})^2 | z_i^{\text{sd}}, z_{i'}^{\text{sd}}\} &= E\{(b_{i,i'} z_{i'}^{\text{sd}} + \sum_{l \neq i, i'} b_{i,l} z_l^{\text{sd}})^2 (b_{i',i} z_i^{\text{sd}} + \sum_{l \neq i, i'} b_{i',l} z_l^{\text{sd}})^2 | z_i^{\text{sd}}, z_{i'}^{\text{sd}}\} \\
 &= E\{(N_1 + N_2 + N_3)(N'_1 + N'_2 + N'_3) | z_i^{\text{sd}}, z_{i'}^{\text{sd}}\} \\
 &= E\{N_1 N'_1 | z_i^{\text{sd}}, z_{i'}^{\text{sd}}\} + E\{N_1 N'_2 | z_i^{\text{sd}}, z_{i'}^{\text{sd}}\} + E\{N_1 N'_3 | z_i^{\text{sd}}, z_{i'}^{\text{sd}}\} \\
 & \quad + E\{N_2 N'_1 | z_i^{\text{sd}}, z_{i'}^{\text{sd}}\} + E\{N_2 N'_2 | z_i^{\text{sd}}, z_{i'}^{\text{sd}}\} + E\{N_2 N'_3 | z_i^{\text{sd}}, z_{i'}^{\text{sd}}\} \\
 & \quad + E\{N_3 N'_1 | z_i^{\text{sd}}, z_{i'}^{\text{sd}}\} + E\{N_3 N'_2 | z_i^{\text{sd}}, z_{i'}^{\text{sd}}\} + E\{N_3 N'_3 | z_i^{\text{sd}}, z_{i'}^{\text{sd}}\},
 \end{aligned}$$

where $N_1 = (\sum_{l \neq i, i'} b_{i,l} z_l^{\text{sd}})^2$, $N_2 = 2 \sum_{l \neq i, i'} b_{i,l} z_l^{\text{sd}} b_{i,i'} z_{i'}^{\text{sd}}$, $N_3 = b_{i,i'}^2 (z_{i'}^{\text{sd}})^2$, $N'_1 = (\sum_{l \neq i, i'} b_{i',l} z_l^{\text{sd}})^2$, $N'_2 = 2 \sum_{l \neq i, i'} b_{i',l} z_l^{\text{sd}} b_{i,i'} z_{i'}^{\text{sd}}$, $N'_3 = b_{i',i}^2 (z_i^{\text{sd}})^2$. By direct calculations, it is easy to see that

$$\begin{aligned}
 E\{N_1 N'_1 | z_i^{\text{sd}}, z_{i'}^{\text{sd}}\} &= M_4 \sum_{l \neq i, i'} b_{i,l}^2 b_{i',l}^2 + \sum_{\substack{l_1, l_2 \neq i, i' \\ l_1 \neq l_2}} b_{i,l_1}^2 b_{i',l_2}^2 + 2 \sum_{\substack{l_1, l_2 \neq i, i' \\ l_1 \neq l_2}} b_{i,l_1} b_{i,l_2} b_{i',l_1} b_{i',l_2}, \\
 E\{N_1 N'_2 | z_i^{\text{sd}}, z_{i'}^{\text{sd}}\} &= 2M_3 b_{i,i'} z_{i'}^{\text{sd}} \sum_{l \neq i, i'} b_{i,l}^2 b_{i',l}, \\
 E\{N_1 N'_3 | z_i^{\text{sd}}, z_{i'}^{\text{sd}}\} &= b_{i',i}^2 (z_i^{\text{sd}})^2 \sum_{l \neq i, i'} b_{i,l}^2, \\
 E\{N_2 N'_1 | z_i^{\text{sd}}, z_{i'}^{\text{sd}}\} &= 2M_3 b_{i,i'} z_{i'}^{\text{sd}} \sum_{l \neq i, i'} b_{i',l}^2 b_{i,l}, \\
 E\{N_2 N'_2 | z_i^{\text{sd}}, z_{i'}^{\text{sd}}\} &= 4b_{i,i'}^2 z_{i'}^{\text{sd}} z_i^{\text{sd}} \sum_{l \neq i, i'} b_{i,l} b_{i',l}, \\
 E\{N_2 N'_3 | z_i^{\text{sd}}, z_{i'}^{\text{sd}}\} &= E\{N_3 N'_2 | z_i^{\text{sd}}, z_{i'}^{\text{sd}}\} = 0, \\
 E\{N_3 N'_1 | z_i^{\text{sd}}, z_{i'}^{\text{sd}}\} &= b_{i,i'}^2 (z_{i'}^{\text{sd}})^2 \sum_{l \neq i, i'} b_{i',l}^2, \\
 E\{N_3 N'_3 | z_i^{\text{sd}}, z_{i'}^{\text{sd}}\} &= b_{i,i'}^2 b_{i',i}^2 (z_i^{\text{sd}})^2 (z_{i'}^{\text{sd}})^2.
 \end{aligned}$$

Therefore, it can be shown that

$$\begin{aligned}
 & \sum_{i \neq i'} [E\{D_i D_{i'}\} - E\{D_i\}E\{D_{i'}\}] \\
 = & 4M_4 \sum_{i \neq i'} \sum_{\substack{l \neq i, i' \\ l_1 \neq l_2}} b_{i,l}^2 b_{i',l}^2 + 4 \sum_{i \neq i'} \sum_{\substack{l_1, l_2 \neq i, i' \\ l_1 \neq l_2}} b_{i,l_1}^2 b_{i',l_2}^2 \\
 & + 8 \sum_{i \neq i'} \sum_{\substack{l_1, l_2 \neq i, i' \\ l_1 \neq l_2}} b_{i,l_1} b_{i,l_2} b_{i',l_1} b_{i',l_2} + 4M_3^2 \sum_{i \neq i'} b_{i,i'} \sum_{l \neq i, i'} b_{i,l}^2 b_{i',l} \\
 & + 4M_3^2 \sum_{i \neq i'} b_{i',i} \sum_{l \neq i, i'} b_{i,l} b_{i',l}^2 + 2(1 + M_4) \sum_{i \neq i'} b_{i,i'}^2 \sum_{l \neq i, i'} b_{i,l}^2 \\
 & + 2(1 + M_4) \sum_{i \neq i'} b_{i,i'}^2 \sum_{l \neq i, i'} b_{i',l}^2 + 4M_3^2 \sum_{i \neq i'} b_{i',i}^2 \sum_{l \neq i, i'} b_{i,l} b_{i',l} \\
 & + (1 + M_4)^2 \sum_{i \neq i'} b_{i,i'}^4 - 4 \sum_{i \neq i'} \sum_{l \neq i} b_{i,l}^2 \sum_{l \neq i'} b_{i',l}^2 \\
 \leq & 4M_4 \sum_{i \neq i'} \sum_{l \neq i, i'} b_{i,l}^2 b_{i',l}^2 + 8 \sum_{i \neq i'} \sum_{\substack{l_1, l_2 \neq i, i' \\ l_1 \neq l_2}} b_{i,l_1} b_{i,l_2} b_{i',l_1} b_{i',l_2} + 4M_3^2 \sum_{i \neq i'} b_{i,i'} \sum_{l \neq i, i'} b_{i,l}^2 b_{i',l} \\
 & + 4M_3^2 \sum_{i \neq i'} b_{i',i} \sum_{l \neq i, i'} b_{i,l} b_{i',l}^2 + 2(1 + M_4) \sum_{i \neq i'} b_{i,i'}^2 \sum_{l \neq i, i'} b_{i,l}^2 \\
 & + 2(1 + M_4) \sum_{i \neq i'} b_{i,i'}^2 \sum_{l \neq i, i'} b_{i',l}^2 + 4M_3^2 \sum_{i \neq i'} b_{i',i}^2 \sum_{l \neq i, i'} b_{i,l} b_{i',l} + (1 + M_4)^2 \sum_{i \neq i'} b_{i,i'}^4 \\
 \leq & (M_4^2 + 12M_3^2 + 10M_4 + 13)\text{trace}(B^4).
 \end{aligned}$$

The last inequality holds because each term in the summation is bounded by $\text{trace}(B^4)$ multiplied by suitable constants.

Since $B = (A - A_0)/s_c$, we have $B^2 \leq 2(A^2 + A_0^2)/s_c^2$ and $B^4 \leq 8(A^4 + A_0^4)/s_c^4$. So it holds that

$$\text{trace}(B^4) \leq 16s_c^{-4}\text{trace}(A^4),$$

where the last inequality follows from the trivial fact $\text{trace}(A^4) \geq \sum_{i=1}^c a_{i,i}^4$. From the above analysis, we get that

$$|J_1| \leq C_2(16M_4^2 + 192M_3^2 + 163M_4 + 213)\psi_c^2 s_c^{-4}\text{trace}(A^4). \quad (54)$$

For J_2 , it holds that

$$\begin{aligned}
 |J_2| &= \left| 2 \sum_{i=1}^c \int_0^1 (1-t) E\{\ddot{g}(Q_2 + t(Q'_2 - Q_2))(\tilde{z}_i^{\text{sd}} - z_i^{\text{sd}})^3 (e_i^T B z^{\text{sd}})^3\} \right| \\
 &\leq 2 \|\ddot{g}\|_{\text{sup}} \sum_{i=1}^c E\{|\tilde{z}_i^{\text{sd}} - z_i^{\text{sd}}|^3 |e_i^T B z^{\text{sd}}|^3\} \\
 &= 2 \|\ddot{g}\|_{\text{sup}} \sum_{i=1}^c E\{|\tilde{z}_i^{\text{sd}} - z_i^{\text{sd}}|\} E\{|e_i^T B z^{\text{sd}}|^3\} \\
 &\leq 32C_3 E\{|z_i^{\text{sd}}|^3\} \psi_c^3 \sum_{i=1}^c E\{|e_i^T B z^{\text{sd}}|^3\} \\
 &\leq 32C_3 E\{|z_i^{\text{sd}}|^3\} \psi_c^3 \sum_{i=1}^c E\{|e_i^T B z^{\text{sd}}|^4\}^{1/2} E\{|e_i^T B z^{\text{sd}}|^2\}^{1/2} \\
 &= 32\sqrt{3}C_3 E\{|z_i^{\text{sd}}|^3\} \psi_c^3 \sum_{i=1}^c (e_i^T B^2 e_i)^{3/2} \\
 &\leq 32\sqrt{3}C_3 E\{|z_i^{\text{sd}}|^3\} \psi_c^3 \sqrt{\sum_{i=1}^c (e_i^T B^2 e_i)^2 \sum_{i=1}^c e_i^T B^2 e_i} \\
 &\leq 32\sqrt{3/2}C_3 E\{|z_i^{\text{sd}}|^3\} \psi_c^3 \sqrt{\text{trace}(B^4)} \\
 &\leq 128\sqrt{3/2}C_3 E\{|z_i^{\text{sd}}|^3\} \psi_c^3 s_c^{-2} \sqrt{\text{trace}(A^4)}.
 \end{aligned}$$

By (47), (54) and $s_c^2 \asymp h^{-1}$, we have

$$\begin{aligned}
 |J_1| &\lesssim C_2(16M_4^2 + 192M_3^2 + 163M_4 + 213)\psi_c^2 h, \\
 |J_2| &\lesssim 128\sqrt{3/2}C_3 E\{|z_i^{\text{sd}}|^3\} \psi_c^3 h^{1/2}.
 \end{aligned}$$

By (49) the following holds uniformly for $u \in \mathbb{R}$:

$$E\{G_u(Q_2)\} - E\{G_u(V)\} \rightarrow 0, \quad c \rightarrow \infty. \quad (55)$$

Similarly, for $\tilde{G}_u(s) = g_0^*(\psi_n(s-u) + 1)$, it can be shown that the following statement holds uniformly for $u \in \mathbb{R}$:

$$E\{\tilde{G}_u(Q_2)\} - E\{\tilde{G}_u(V)\} \rightarrow 0, \quad c \rightarrow \infty. \quad (56)$$

By elementary facts, we have

$$\begin{aligned}
 P(Q_2 \leq u) &\leq E\{G_u(Q_2)\} \leq P(Q_2 \leq u + \psi_c^{-1}), \\
 P(V \leq u) &\leq E\{G_u(V)\} \leq P(V \leq u + \psi_c^{-1}), \\
 P(Q_2 \leq u - \psi_c^{-1}) &\leq E\{\tilde{G}_u(Q_2)\} \leq P(Q_2 \leq u), \\
 P(V \leq u - \psi_c^{-1}) &\leq E\{\tilde{G}_u(V)\} \leq P(V \leq u).
 \end{aligned} \quad (57)$$

By (55), (56) and (57), the following statements hold uniformly for $u \in \mathbb{R}$,

$$\begin{aligned}
 P(Q_2 \leq u) - P(V \leq u) &\leq E\{G_u(Q_2) - G_u(V)\} + P(V \leq u + \psi_c^{-1}) - P(V \leq u) \rightarrow 0, \\
 P(V \leq u) - P(Q_2 \leq u) &\leq E\{\tilde{G}_u(V) - \tilde{G}_u(Q_2)\} + P(V \leq u) - P(V \leq u - \psi_c^{-1}) \rightarrow 0.
 \end{aligned}$$

Hence, as c tends to infinity,

$$\sup_{u \in \mathbb{R}} |P(Q_2 \leq u) - P(V \leq u)| \rightarrow 0.$$

This, together with $Q_1 = o_P(1)$, proves

$$\frac{cT_{\mu^*,t,c}^* - \text{trace}(A)\tau_k^{*2}}{s_c\tau_k^{*2}} \xrightarrow{d} N(0,1), \text{ as } c \rightarrow \infty. \quad (58)$$

Let z_i 's and $T_{\mu,t,c}$ be the quantized samples and testing statistics in Theorem 3, then we have

$$\begin{aligned} \frac{cT_{\mu,t,c} - \sum_{i=1}^c a_{i,i}\widehat{\tau}_k^2}{s_c\widehat{\tau}_k^2} &= \frac{cT_{\mu,t,c} - cT_{\mu^*,t,c}^*}{s_c\widehat{\tau}_k^2} + \frac{cT_{\mu^*,t,c}^* - \sum_{i=1}^c a_{i,i}\widehat{\tau}_k^2}{s_c\widehat{\tau}_k^2} \\ &= R_1 + R_2. \end{aligned}$$

We will analyze these two terms separately. For R_1 , one has that

$$\frac{cT_{\mu,t,c} - cT_{\mu^*,t,c}^*}{s_c\widehat{\tau}_k^2} = \frac{z^T Az - z^{*T} Az^*}{s_c\widehat{\tau}_k^2} = \frac{\Delta^{*T} A \Delta^*}{s_c\widehat{\tau}_k^2} + \frac{2\Delta^{*T} Az^*}{s_c\widehat{\tau}_k^2}, \quad (59)$$

where $\Delta^* = (\Delta_1^*, \dots, \Delta_c^*)^T$ with $\Delta_i^* = z_i - z_i^*$ which satisfies $E(\Delta_i^{*2}) \leq C_k^2(t) + 1/n$ under Condition **B**.

For the first term, since $\|\Delta^*\|^2 \leq O_p(cC_k(t)^2 + cn^{-1})$ and $A \leq I_c$, it follows that

$$\frac{\Delta^{*T} A \Delta^*}{s_c\widehat{\tau}_k^2} \leq \frac{\|\Delta^*\|^2}{s_c\widehat{\tau}_k^2} \leq \frac{c}{s_c\widehat{\tau}_k^2} O_p(C_k(t)^2 + n^{-1}) = O_p\left(\frac{cC_k(t)^2}{h^{-1/2}c/n}\right) = O_p\left(nh^{1/2}C_k(t)^2\right) = o_p(1),$$

where the last equality follows from the condition $b \gg \log_2\left(\sqrt{n\mathcal{T}_n h^{1/2}}\right)$, which implies that

$$C_k(t)^2 \asymp \frac{\mathcal{T}_n}{2^{2b}} \ll \frac{\mathcal{T}_n}{(nh^{1/2})\mathcal{T}_n} = (nh^{1/2})^{-1}.$$

For the second term in (59), using the fact that $(\Delta_i^*, z_i^*)^T, (\Delta_j^*, z_j^*)^T$ are independent if $i \neq j$, and $Ez^* = 0$, it is straightforward to show that

$$\begin{aligned}
 E((\Delta^* - E\Delta^*)^T Az^*)^2 &= \sum_{i=1}^c a_{ii}^2 E((\Delta_i^* - E\Delta_i^*)z_i^*)^2 + \sum_{i=1}^c \sum_{j=1}^c a_{ij}^2 E((\Delta_i^* - E\Delta_i^*)(\Delta_i^* - E\Delta_i^*)z_j^*z_j^*) \\
 &\quad + \sum_{i=1}^c \sum_{j=1}^c a_{ij}a_{ji} E((\Delta_i^* - E\Delta_i^*)z_j^*(\Delta_j^* - E\Delta_j^*)z_i^*) \\
 &\quad + \sum_{i=1}^c \sum_{j=1}^c a_{ii}a_{jj} E((\Delta_i^* - E\Delta_i^*)z_i^*(\Delta_j^* - E\Delta_j^*)z_j^*) \\
 &\quad + \sum_{i=1}^c \sum_{j=1}^c a_{jj}a_{ii} E((\Delta_j^* - E\Delta_j^*)z_j^*(\Delta_i^* - E\Delta_i^*)z_i^*) \\
 &\leq \sum_{i=1}^c a_{ii}^2 C_k(t)^2 \tau_k^{*2} + \sum_{i=1}^c \sum_{j=1}^c a_{ij}^2 C_k(t)^2 \tau_k^{*2} \\
 &\quad + \sum_{i=1}^c \sum_{j=1}^c a_{ij}a_{ji} C_k(t)^2 \tau_k^{*2} + \sum_{i=1}^c \sum_{j=1}^c a_{ii}a_{jj} C_k(t)^2 \tau_k^{*2} + \sum_{i=1}^c \sum_{j=1}^c a_{jj}a_{ii} C_k(t)^2 \tau_k^{*2} \\
 &= \left(\sum_{i=1}^c a_{ii}^2 \right) C_k(t)^2 \tau_k^{*2} + 2\text{trace}(A^2) C_k(t)^2 \tau_k^{*2} + 2[\text{trace}(A)]^2 C_k(t)^2 \tau_k^{*2}.
 \end{aligned}$$

In the proof of (47), we have shown that $a_{ii} \asymp (ch)^{-1}$, $\text{trace}(A) \asymp \text{trace}(A^2) \asymp h^{-1}$, thence we have that

$$E((\Delta^* - E\Delta^*)^T Az^*)^2 \lesssim h^{-2} C_k(t)^2 \tau_k^{*2},$$

which implies that

$$\frac{2(\Delta^* - E\Delta^*)^T Az^*}{s_c \widehat{\tau}_k^2} = O_p \left(\frac{h^{-1} C_k(t) \tau_k^*}{s_c \widehat{\tau}_k^2} \right) = O_p \left(\frac{h^{-1} C_k(t) \sqrt{c/n}}{h^{-1/2} c/n} \right) = O_p \left(C_k(t) \sqrt{\frac{n}{ch}} \right).$$

Furthermore, since $A \leq I_c$, we have that

$$E[(E\Delta^*)^T Az^*]^2 = E[(E\Delta^*)^T Az^* z^{*T} A (E\Delta^*)] = \tau_k^{*2} (E\Delta^*)^T A^2 (E\Delta^*) \leq \tau_k^{*2} \|E\Delta^*\|^2 \leq c C_k(t)^2 \tau_k^{*2},$$

which implies that

$$\frac{2(E\Delta^*)^T Az^*}{s_c \widehat{\tau}_k^2} = O_p \left(\frac{\sqrt{c} C_k(t) \tau_k^*}{s_c \widehat{\tau}_k^2} \right) = O_p \left(\frac{\sqrt{c} C_k(t) \sqrt{c/n}}{h^{-1/2} c/n} \right) = O_p \left(C_k(t) \sqrt{nh} \right),$$

and consequently,

$$\frac{2\Delta^{*T} Az^*}{s_c \widehat{\tau}_k^2} = O_p \left(C_k(t) \sqrt{\frac{n}{ch}} + C_k(t) \sqrt{nh} \right). \quad (60)$$

Using the condition $b \gg \log_2 \left(\sqrt{(nh^{1/2} + n(ch)^{-1}) \mathcal{T}_n} \right)$, and the fact that $h \rightarrow 0$, one has that

$$C_k(t)^2 \asymp \frac{\mathcal{T}_n}{2^{2b}} \ll \frac{\mathcal{T}_n}{(nh^{1/2} + n(ch)^{-1}) \mathcal{T}_n} \leq \frac{\mathcal{T}_n}{(nh + n(ch)^{-1}) \mathcal{T}_n} = (nh + n(ch)^{-1})^{-1},$$

which gives that $\frac{2\Delta^{*T}Az^*}{s_c\widehat{\tau}_k^2} = o_p(1)$. Plugging this back to equation (59), we have that $R_1 = o_p(1)$.

Now we analyze R_2 , by Lemma 11, we have

$$\begin{aligned} \frac{cT_{\mu^*,t,c}^* - \sum_{i=1}^c a_{i,i}\widehat{\tau}_k^2}{s_c\widehat{\tau}_k^2} &= \frac{cT_{\mu^*,t,c}^* - \text{trace}(A)\tau_k^{*2} + \text{trace}(A)\tau_k^{*2} - \text{trace}(A)\widehat{\tau}_k^2}{s_c\tau_k^{*2}} \frac{\tau_k^{*2}}{\widehat{\tau}_k^2} \\ &= \frac{cT_{\mu^*,t,c}^* - \text{trace}(A)\tau_k^{*2}}{s_c\tau_k^{*2}} + \frac{\text{trace}(A)(\tau_k^{*2} - \widehat{\tau}_k^2)}{s_c\tau_k^{*2}} + o_p(1) \\ &= \frac{cT_{\mu^*,t,c}^* - \text{trace}(A)\tau_k^{*2}}{s_c\tau_k^{*2}} + O_p(C_k(t)h^{-1/2}). \end{aligned}$$

Since $k \gg \sqrt{(nh^{1/2} + n(ch)^{-1})\mathcal{T}_n}$, one has that

$$C_k h^{-1/2} \ll \frac{\sqrt{\mathcal{T}_n}}{\sqrt{(nh^{1/2} + \frac{n}{ch})\mathcal{T}_n h}} = \frac{1}{\sqrt{(nh^{3/2} + \frac{n}{c})}} \leq \sqrt{c/n} = O(1).$$

From (58), we get the desired result. ■

Proof of Proposition 4: Suppose $p_\sigma(\cdot)$ is the density of $\sigma\epsilon_1$. By direct calculations, we have

$$E([\widetilde{n}^{-1} \sum_{j=1}^{\widetilde{n}} Q(\epsilon_j)]^4) = 3\sigma^4 \left(\frac{1}{\widetilde{n}^2} - \frac{1}{\widetilde{n}^3} \right) E^2\{Q(\sigma\epsilon_1)^2\} + \frac{\sigma^2 E\{Q(\sigma\epsilon_1)^4\}}{\widetilde{n}^3}. \quad (61)$$

For the first term, under Condition (B), we know $E\{Q(\sigma\epsilon_1)^2\} = O(1)$, which implies $3\sigma^4 \left(\frac{1}{\widetilde{n}^2} - \frac{1}{\widetilde{n}^3} \right) E^2\{Q(\sigma\epsilon_1)^2\} = O(c^2 n^{-2})$. For the second term, we have that

$$\begin{aligned} E\{Q(\sigma\epsilon_1)^4\} &= \sum_{j=2}^{k-1} \mu_j^4 P(\sigma\epsilon_1 \in R_j) + \mu_1^4 P(\sigma\epsilon_1 \in R_1) + \mu_k^4 P(\sigma\epsilon_1 \in R_k) \\ &= \sum_{j=2}^{k-1} \int_{R_j} \mu_j^4 p_\sigma(x) dx + \mu_1^4 P(\sigma\epsilon_1 \in R_1) + \mu_k^4 P(\sigma\epsilon_1 \in R_k) \\ &\leq \sum_{j=2}^{k-1} \int_{R_j} (|x| + C_k(t))^4 p_\sigma(x) dx + \mu_1^4 P(\sigma\epsilon_1 \in R_1) + \mu_k^4 P(\sigma\epsilon_1 \in R_k) \\ &\leq 8 \sum_{j=1}^k \int_{R_j} \sigma^4 x^4 p_\sigma(x) dx + 8C_k(t)^4 + \mu_1^4 P(\sigma\epsilon_1 \in R_1) + \mu_k^4 P(\sigma\epsilon_1 \in R_k) \\ &= 8\sigma^4 E\{\epsilon_1^4\} + 8C_k(t)^4 + \mu_1^4 P(\sigma\epsilon_1 \in R_1) + \mu_k^4 P(\sigma\epsilon_1 \in R_k). \end{aligned}$$

Since $C_k(t)^4 = o(1)$, $E\{\epsilon_1^4\} = O(nc^{-1})$ and $\mu_j^4 P(\sigma\epsilon_1 \in R_j(t)) = O(nc^{-1})$ for $j = 1, k$, one has that $\frac{\sigma^2 E\{Q(\sigma\epsilon_1)^4\}}{\widetilde{n}^3} = O(c^2 n^{-2})$. Plugging this back to equation (61), we get the desired result. ■

Proof of Theorem 5: Without loss of generality, we only consider the case $g_*(x) = 0$ in (2). By Condition **(B)**, we have that $\min\{t_1^2, t_{k-1}^2\} > 4c_s^2\rho^2$, as $n \rightarrow \infty$. Consider the following event:

$$\mathcal{E}_1 = \{\sigma|\epsilon_i| + c_s\rho \leq \sqrt{\mathcal{T}_n} \text{ for all } 1 \leq i \leq n\}. \quad (62)$$

It is easy to show that $P(\mathcal{E}_1) \rightarrow 1$ as $n \rightarrow \infty$ under Condition **(B)**. Thus, we choose N'_η s.t. $P(\mathcal{E}_1) \geq 1 - \eta/3$ if $c \geq N'_\eta$.

Throughout the proof, we suppose that $g \in S_\rho^m(\mathbb{I})$ is the function that generates the samples and f is the integral function of g defined in (15). Let $\omega_i = \tilde{z}_i - \tilde{z}_i^0$, $\omega = (\omega_1, \dots, \omega_c)^T$. It is straightforward to see that $\omega_i = z_i - z_i^0$ under event \mathcal{E}_1 . Because

$$C_k(t)^2 \asymp \frac{\mathcal{T}_n}{2^{2b}} \ll \frac{\mathcal{T}_n}{(nh^{1/2} + n(ch)^{-1})\mathcal{T}_n} \leq \frac{\mathcal{T}_n}{(nh + n(ch)^{-1})\mathcal{T}_n} = (nh + n(ch)^{-1})^{-1} \ll \tau_k^2,$$

it follows by Lemma 14 that there exists N'' s.t., when $c \geq N''$, the following equation holds

$$\sup_{g \in S_\rho^m(\mathbb{I})} \frac{E\{|\tilde{z} - \tilde{z}^0\|^T A z^0\|^2\}}{(1 + (ch^2)^{-1})\tau_k^2 \sum_{i=1}^c (|f(i/c)| + 4C_k(t) + \zeta)^2} \leq 8, \text{ as } c \rightarrow \infty. \quad (63)$$

Consider the event

$$\mathcal{E}_2 = \left\{ |\omega^T A z^0| \leq C'_\eta \sqrt{1 + (ch^2)^{-1}} \tau_k \sqrt{\sum_{i=1}^c (|f(i/c)| + 4C_k(t) + \zeta)^2} \right\},$$

where $C'_\eta = \sqrt{24/\eta}$.

Then

$$1 - P(\mathcal{E}_2) \leq \frac{E\{|\omega^T A z^0\|^2\}}{(C'_\eta)^2 (1 + (ch^2)^{-1}) \tau_k^2 \sum_{i=1}^c (|f(i/c)| + 4C_k(t) + \zeta)^2} \leq \eta/3,$$

which implies that $P(\mathcal{E}_2) \geq 1 - \eta/3$.

Let $\hat{\tau}_{k,0}^2$ be the estimated variance under the null. Then one has that

$$\begin{aligned} \frac{(z^0)^T A z^0 - \text{trace}(A) \hat{\tau}_k^2}{s_c \hat{\tau}_k^2} &= \frac{(z^0)^T A z^0 - \text{trace}(A) \hat{\tau}_{k,0}^2 + \text{trace}(A) \hat{\tau}_{k,0}^2 - \text{trace}(A) \hat{\tau}_k^2}{s_c \hat{\tau}_{k,0}^2} \frac{\hat{\tau}_{k,0}^2}{\hat{\tau}_k^2} \\ &= \frac{(z^0)^T A z^0 - \text{trace}(A) \hat{\tau}_{k,0}^2}{s_c \hat{\tau}_{k,0}^2} + \frac{\text{trace}(A) (\hat{\tau}_{k,0}^2 - \hat{\tau}_k^2)}{s_c \hat{\tau}_{k,0}^2} + o_p(1) \\ &= \frac{(z^0)^T A z^0 - \text{trace}(A) \hat{\tau}_{k,0}^2}{s_c \hat{\tau}_{k,0}^2} + O_p(C_k(t) h^{-1/2}). \end{aligned}$$

Since $k \gg \sqrt{(nh^{1/2} + n(ch)^{-1})\mathcal{T}_n}$, one has that

$$C_k h^{-1/2} \ll \frac{\sqrt{\mathcal{T}_n}}{\sqrt{(nh^{1/2} + \frac{n}{ch})\mathcal{T}_n} h} = \frac{1}{\sqrt{(nh^{3/2} + \frac{n}{c})}} \leq \sqrt{c/n} = O(1).$$

It follows from Theorem 3 that

$$\frac{(z^0)^T A z^0 - \text{trace}(A) \widehat{\tau}_k^2}{s_c \widehat{\tau}_k^2} = O_P(1).$$

Hence, there exists $C''_\eta > 0$ s.t. $P(\mathcal{E}_3) \geq 1 - \eta/3$ for all $c \geq N'_\eta$ and N'' , where

$$\mathcal{E}_3 = \left\{ \left| \frac{(z^0)^T A z^0 - \text{trace}(A) \widehat{\tau}_k^2}{s_c \widehat{\tau}_k^2} \right| \leq C''_\eta \right\}.$$

Let $\mathcal{E} = \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3$, then $P(\mathcal{E}) \geq 1 - \eta$ for any $c \geq N'_\eta$ and N'' .

Suppose $g \in S_\rho^m(\mathbb{I})$ satisfies $\|g\|_c \geq C_\eta \delta_*$, where

$$C_\eta = \max \{ 6\varrho\rho^2, 384, (72C'_\eta)^2, 6(C''_\eta + z_{1-\alpha/2} + 1) \}, \quad (64)$$

$$\delta_* = \sqrt{c^{-1} \tau_k^2 (1 + s_c + (ch^2)^{-1}) + \lambda + c^{-2m} + C_k(t)^2 + \zeta^2}, \quad (65)$$

where $\tau_k^2 = O(\tilde{n}^{-1})$, $\zeta = \max_{i=1, \dots, c} \left| f(i/c) - \frac{1}{\tilde{n}} \sum_{j=(i-1)\tilde{n}+1}^{i\tilde{n}} g(j/n) \right| = O(n^{-1})$.

It follows from Lemma 13 that, on \mathcal{E} , $|\omega_i - f(i/c)| \leq 4C_k(t) + \zeta$. Since $A \leq I_c$, we get that

$$(\omega - \mathbf{f})^T A (\omega - \mathbf{f}) \leq \sum_{i=1}^c (\omega_i - f(i/c))^2 \leq 32cC_k(t)^2 + 2c\zeta^2,$$

which, together with Lemma 16, leads to that

$$\begin{aligned} \omega^T A \omega &\geq \frac{1}{2} \mathbf{f}^T A \mathbf{f} - (\omega - \mathbf{f})^T A (\omega - \mathbf{f}) \\ &= \frac{c}{2} \|g\|_c^2 - \frac{1}{2} \mathbf{f}^T (I_c - A) \mathbf{f} - (\omega - \mathbf{f})^T A (\omega - \mathbf{f}) \\ &\geq \frac{c}{2} \|g\|_c^2 - \frac{1}{2} \varrho c (\lambda + c^{-2m}) \rho^2 - 32cC_k(t)^2 - 2c\zeta^2. \end{aligned}$$

Therefore, on \mathcal{E} , we have

$$\begin{aligned}
 & \frac{cT_{\mu,t,c} - \text{trace}(A)\widehat{\tau}_k^2}{s_c\widehat{\tau}_k^2} \\
 = & \frac{z^T Az - (z^0)^T Az^0}{s_c\widehat{\tau}_k^2} + \frac{(z^0)^T Az^0 - \text{trace}(A)\widehat{\tau}_k^2}{s_c\widehat{\tau}_k^2} \\
 \geq & \frac{z^T Az - (z^0)^T Az^0}{s_c\widehat{\tau}_k^2} - C''_\eta \\
 = & \frac{\omega^T A\omega + 2\omega^T Az^0}{s_c\widehat{\tau}_k^2} - C''_\eta \\
 \geq & \frac{\frac{c}{2}\|g\|_c^2 - \frac{1}{2}\varrho c(\lambda + c^{-2m})\rho^2 - 32cC_k(t)^2 - 2c\zeta^2 - 2C'_\eta\tau_k\sqrt{(1+\frac{1}{ch^2})\sum_{i=1}^c(|f(i/c)|+4C_k(t)+\zeta)^2}}{s_c\widehat{\tau}_k^2} - C''_\eta \\
 \geq & \frac{\frac{c}{2}\|g\|_c^2 - \frac{1}{2}\varrho c(\lambda + c^{-2m})\rho^2 - 32cC_k(t)^2 - 2c\zeta^2 - 6C'_\eta\sqrt{1+(ch^2)^{-1}}\tau_k\sqrt{c}\|g\|_c}{s_c\widehat{\tau}_k^2} - C''_\eta \tag{66} \\
 = & \frac{\frac{c}{2}\|g\|_c^2 \left(1 - \frac{\frac{1}{2}\varrho c(\lambda + c^{-2m})\rho^2}{\frac{c}{2}\|g\|_c^2} - \frac{32cC_k(t)^2}{\frac{c}{2}\|g\|_c^2} - \frac{2c\zeta^2}{\frac{c}{2}\|g\|_c^2} - \frac{6C'_\eta\sqrt{1+(ch^2)^{-1}}\tau_k\sqrt{c}\|g\|_c}{\frac{c}{2}\|g\|_c^2}\right)}{s_c\widehat{\tau}_k^2} - C''_\eta \\
 \geq & \frac{\frac{c}{6}\|g\|_c^2}{s_c\widehat{\tau}_k^2} - C''_\eta > z_{1-\alpha/2}, \tag{67}
 \end{aligned}$$

where (66) follows from $C_\eta > 12$ (see (64)), i.e.,

$$\sum_{i=1}^c f(i/c)^2 = c\|g\|_c^2 \geq cC_\eta\delta_*^2 \geq 48cC_k(t)^2, \quad \sum_{i=1}^c f(i/c)^2 = c\|g\|_c^2 \geq cC_\eta\delta_*^2 \geq 3c\zeta^2,$$

which leads to

$$\sum_{i=1}^c (|f(i/c)| + 4C_k(t) + \zeta)^2 \leq 3 \sum_{i=1}^c f(i/c)^2 + 48cC_k(t)^2 + 3c\zeta^2 \leq 9 \sum_{i=1}^c f(i/c)^2 = 9c\|g\|_c^2;$$

and (67) follows from (64), i.e.,

$$\begin{aligned}
 \frac{\frac{1}{2}\varrho c(\lambda + c^{-2m})\rho^2}{\frac{c}{2}\|g\|_c^2} & \leq 1/6, \\
 \frac{32cC_k(t)^2}{\frac{c}{2}\|g\|_c^2} & \leq 1/6, \\
 \frac{2c\zeta^2}{\frac{c}{2}\|g\|_c^2} & \leq 1/6, \\
 \frac{6C'_\eta\sqrt{1+(ch^2)^{-1}}\tau_k\sqrt{c}\|g\|_c}{\frac{c}{2}\|g\|_c^2} & \leq 1/6.
 \end{aligned}$$

Then for any $g \in S_\rho^m(\mathbb{I})$ satisfying $\|g\|_c \geq C_\eta \delta_*$, where C_η, δ_* are defined in (64) (65), there exist $N_\eta \equiv \max\{N'_\eta, N''\}$ such that for any $c \geq N_\eta$, we have

$$\begin{aligned} & P(\text{reject } H_0 | H_1 \text{ is true}) \\ & \geq P\left(\mathcal{E} \text{ and } \left| \frac{cT_{\mu,t,c} - \text{trace}(A)\widehat{\tau}_k^2}{s_c \widehat{\tau}_k^2} \right| \geq z_{1-\alpha/2}\right) \\ & = P(\mathcal{E}) \geq 1 - \eta. \end{aligned}$$

In the end, since $h = \lambda^{1/(2m)}, nh^{1/2}C_k(t)^2 = o(1), ch \rightarrow \infty, \zeta = O(n^{-1})$, immediately, one has that $\|g\|_c \geq C_\eta \delta_*$ is equivalent as $\|g\|_c \geq C_\eta \delta_{n,c,\lambda}$. This proves the desired result. \blacksquare

Proof of Theorem 6: Suppose $g = \beta x + \alpha$ is the “true” function under H_0^{linear} and $y_i = g(i/n) + \sigma \epsilon_i, i = 1, \dots, n$. We use \widehat{g} to denote the least-square estimator of g based on $Q(y_i)$'s. Consider the following two events:

$$\begin{aligned} \mathcal{E}_1 &= \{\sigma|\epsilon_i| + c_s \rho \leq \sqrt{\mathcal{T}_n} \text{ for all } 1 \leq i \leq n\}, \\ \mathcal{E}_2 &= \{|g(i/n) - \widehat{g}(i/n)| \leq \sqrt{\mathcal{T}_n} \text{ for all } 1 \leq i \leq n\}. \end{aligned}$$

It is easy to show that $P(\mathcal{E}_1 \cap \mathcal{E}_2) \rightarrow 1$, as $n \rightarrow \infty$. Since $\min\{t_1^2, t_{k-1}^2\} = \mathcal{T}_n > 4c_s^2 \rho^2$ as $n \rightarrow \infty$, under event $\mathcal{E}_1 \cap \mathcal{E}_2$, for $j = 1, \dots, c$, one has that

$$\sigma|\epsilon_j| \leq \min\{|t_1|, |t_{k-1}|\}, \quad |y_j| \leq \min\{|t_1|, |t_{k-1}|\}, \quad |g(j/n) - \widehat{y}_j| \leq \min\{|t_1|, |t_{k-1}|\}.$$

Furthermore, we have

$$\begin{aligned} z_i^{\text{linear}} &\leq \frac{\sum_{j=(i-1)\widetilde{n}+1}^{i\widetilde{n}} Q(y_j^{\text{linear}})}{\widetilde{n}} + C_k(t) \\ &= \frac{\sum_{j=(i-1)\widetilde{n}+1}^{i\widetilde{n}} Q(Q(y_j) - \widehat{y}_j)}{\widetilde{n}} + C_k(t) \\ &= \frac{\sum_{j=(i-1)\widetilde{n}+1}^{i\widetilde{n}} Q(y_j - g(j/n) + Q(y_j) - y_j + g(j/n) - \widehat{y}_j)}{\widetilde{n}} + C_k(t) \\ &\leq \frac{\sum_{j=(i-1)\widetilde{n}+1}^{i\widetilde{n}} Q(y_j - g(j/n)) + Q(Q(y_j) - y_j) + Q(g(j/n) - \widehat{y}_j)}{\widetilde{n}} + 7C_k(t) \\ &\leq \frac{\sum_{j=(i-1)\widetilde{n}+1}^{i\widetilde{n}} Q(\sigma \epsilon_j)}{\widetilde{n}} + \frac{\sum_{j=(i-1)\widetilde{n}+1}^{i\widetilde{n}} Q(g(j/n) - \widehat{y}_j)}{\widetilde{n}} + 9C_k(t) \\ &\leq z_i^0 + \varsigma_i + 10C_k(t), \end{aligned} \tag{68}$$

where $\varsigma_i = \frac{\sum_{j=(i-1)\widetilde{n}+1}^{i\widetilde{n}} Q(g_0(j/n) - \widehat{y}_j)}{\widetilde{n}}$ satisfying $\varsigma_i = O_p(1/n + C_k(t))$, and equations (68), (69) follow from (6), (33). Let $z^0 = (z_1^0, \dots, z_c^0)^T, \varsigma = (\varsigma_1, \dots, \varsigma_c)^T$. Therefore, the test statistic

$$\begin{aligned} cT_{\text{linear}} &= c\|\widehat{g}_{\text{linear}, \mu, t, c}^{\text{B}}\|^2 = (z^{\text{linear}})^T A z^{\text{linear}} \\ &\leq (z^0)^T A z^0 + \varsigma^T A \varsigma + 100\overrightarrow{C_k(t)}^T A \overrightarrow{C_k(t)} + 2((z^0)^T A \varsigma + 10(z^0)^T A \overrightarrow{C_k(t)} + 10\varsigma^T A \overrightarrow{C_k(t)}) \\ &= T_1 + T_2 + T_3 + T_4, \end{aligned}$$

where $\overrightarrow{C_k(t)} = (C_k(t), \dots, C_k(t))^T$. Now we proceed to prove that cT_{linear} is dominated by T_1 . Using the fact that z_i^0, z_j^0 are independent of each other if $i \neq j$, and $E(\{z_i^0\}^2) = O(\tau_k^2) = O(c/n)$, $E(\{z_i^0\}^4) = O(c^2/n^2)$, then for the first term T_1 , it is straightforward to show that

$$\begin{aligned} E((z^0)^T A z^0)^2 &= \sum_{i=1}^c a_{ii}^2 E(z_i^0)^4 + \sum_{i=1}^c \sum_{j=1}^c a_{ij}^2 E(z_i^0 z_i^0 z_j^0 z_j^0) + \sum_{i=1}^c \sum_{j=1}^c a_{ij} a_{ji} E(z_i^0 z_j^0 z_j^0 z_i^0) \\ &\quad + \sum_{i=1}^c \sum_{j=1}^c a_{ii} a_{jj} E(z_i^0 z_i^0 z_j^0 z_j^0) + \sum_{i=1}^c \sum_{j=1}^c a_{jj} a_{ii} E(z_j^0 z_j^0 z_i^0 z_i^0) \\ &\asymp c^2 n^{-2} \left(\sum_{i=1}^c a_{ii}^2 + \sum_{i=1}^c \sum_{j=1}^c a_{ij}^2 + \sum_{i=1}^c \sum_{j=1}^c a_{ij} a_{ji} + \sum_{i=1}^c \sum_{j=1}^c a_{ii} a_{jj} + \sum_{i=1}^c \sum_{j=1}^c a_{jj} a_{ii} \right) \\ &= c^2 n^{-2} \left(\sum_{i=1}^c a_{ii}^2 + 2\text{trace}(A^2) + 2[\text{trace}(A)]^2 \right). \end{aligned}$$

In the proof to achieve equation (47), we have shown that $a_{ii} \asymp (ch)^{-1}$, $\text{trace}(A) \asymp \text{trace}(A^2) \asymp h^{-1}$, thence we have that

$$E((z^0)^T A z^0)^2 \asymp c^2 (nh)^{-2}. \quad (70)$$

Furthermore, since $A \leq I_c$, we have that

$$\varsigma^T A \varsigma \leq \sum_{i=1}^c \varsigma_i^2 = \sum_{i=1}^c \left[\frac{\sum_{j=(i-1)\tilde{n}+1}^{i\tilde{n}} Q(g_0(j/n) - \hat{y}_j)}{\tilde{n}} \right]^2 = O_p(c/n + cC_k(t)^2),$$

$$\overrightarrow{C_k(t)}^T A \overrightarrow{C_k(t)} \leq cC_k(t)^2.$$

Since $C_k(t)^2 \asymp \frac{T_n}{2^{2b}} \ll \frac{T_n}{(nh^{1/2} + n(ch)^{-1})T_n} \leq \frac{T_n}{(nh + n(ch)^{-1})T_n} = (nh + n(ch)^{-1})^{-1}$, one has that

$$cC_k(t)^2 \ll c(nh + n(ch)^{-1})^{-1} \leq c(nh)^{-1}.$$

Together with the fact that $cn^{-1} \ll c(nh)^{-1}$ and equation (70), one has that

$$T_2 \ll T_1, T_3 \ll T_1.$$

Similarly, it can be shown that $T_4 \ll T_1$. Therefore, $cT_{\text{linear}} \asymp T_1$. The dominated term T_1 in cT_{linear} is nothing but $cT_{\mu,t,c}$ for testing $H_0 : g_0(x) = 0$ based on z^0 . Therefore, in keep with Lemma 11, the limiting distribution of T_{linear} under H_0^{linear} should have the same limiting distribution as $T_{\mu,t,c}$ under $H_0 : g_0(x) = 0$. Thus, according to Theorem 3 and Lemma 12, the result is proved. \blacksquare

Proof of Theorem 7: The proof of Theorem 7 is similar to Theorem 5. Let g be the function which generates the observations and $\mathcal{P}_{\mathcal{L}(\mathbb{I})}(g) = \arg \min_{f \in \mathcal{L}(\mathbb{I})} \|g - f\|^2$ be the

projection of $g(\cdot)$ to $\mathcal{L}(\mathbb{I})$. We further define $f(x)$ be the integral function associated with $g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)$, as defined in (15), that is,

$$f(x) = \frac{1}{2\Delta} \int_{\max(x-\Delta,0)}^{\min(x+\Delta,1)} [g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)](s) ds, \text{ for } x \in [0, 1], \Delta = \frac{1}{c}.$$

Therefore, $\|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c = \sqrt{\sum_{i=1}^c f^2(i/c)/c}$. To proceed, we first define \widehat{g}^* be the least squared estimator based on $Q(\mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)(j/n) + \sigma\epsilon_j)$ which satisfies $|\widehat{g}(i/n) - \widehat{g}^*(i/n)| \leq C_k(t)$. Let $z_{i,0}^{\text{linear}}$ be the z_i^{linear} based on $\mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)$. According to (6), onw has that

$$\left| z_{i,0}^{\text{linear}} - \widetilde{n}^{-1} \sum_{j=(i-1)\widetilde{n}+1}^{\widetilde{i}\widetilde{n}} Q\left(Q(\mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)(j/n) + \sigma\epsilon_j) - \widehat{g}^*(j/n)\right) \right| \leq C_k(t), \text{ for } i = 1, \dots, c.$$

Let $z_0^{\text{linear}} = (z_{1,0}^{\text{linear}}, \dots, z_{c,0}^{\text{linear}})^T$. Before proceeding, we first define some notations to ease the calculations. Define

$$\begin{aligned} \widetilde{z}_{i,0}^{\text{linear}} &= z_{i,0}^{\text{linear}} \mathbb{1}\left(\max\{\sigma|\epsilon_j| + c_s\rho, |g(j/n) - \widehat{g}(j/n)|, |\mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)(j/n) - \widehat{g}^*(j/n)|\} \leq \sqrt{\mathcal{T}_n} \text{ for all } j = (i-1)\widetilde{n}+1, \dots, \widetilde{i}\widetilde{n}\right), \\ \widetilde{z}_i^{\text{linear}} &= z_i^{\text{linear}} \mathbb{1}\left(\max\{\sigma|\epsilon_j| + c_s\rho, |g(j/n) - \widehat{g}(j/n)|, |\mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)(j/n) - \widehat{g}^*(j/n)|\} \leq \sqrt{\mathcal{T}_n} \text{ for all } j = (i-1)\widetilde{n}+1, \dots, \widetilde{i}\widetilde{n}\right). \end{aligned}$$

Similar to Lemma 13, we want to find an upper bound of $|\widetilde{z}_i^{\text{linear}} - \widetilde{z}_{i,0}^{\text{linear}}|$. It is straightforward to show that

$$\begin{aligned} & |\widetilde{z}_i^{\text{linear}} - \widetilde{z}_{i,0}^{\text{linear}}| \tag{71} \\ & \leq \left| \widetilde{n}^{-1} \sum_{j=(i-1)\widetilde{n}+1}^{\widetilde{i}\widetilde{n}} Q(y_j^{\text{linear}}) - Q\left(Q(\mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)(j/n) + \sigma\epsilon_j) - \widehat{g}^*(j/n)\right) \right| + 2C_k(t) \\ & = \left| \widetilde{n}^{-1} \sum_{j=(i-1)\widetilde{n}+1}^{\widetilde{i}\widetilde{n}} Q(Q(y_j) - \widehat{g}(j/n)) - Q\left(Q(\mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)(j/n) + \sigma\epsilon_j) - \widehat{g}^*(j/n)\right) \right| + 2C_k(t) \\ & \leq \left| \widetilde{n}^{-1} \sum_{j=(i-1)\widetilde{n}+1}^{\widetilde{i}\widetilde{n}} Q(y_j) - \widehat{g}(j/n) - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)(j/n) - \sigma\epsilon_j + \widehat{g}^*(j/n) \right| + 6C_k(t) \\ & \leq \left| \widetilde{n}^{-1} \sum_{j=(i-1)\widetilde{n}+1}^{\widetilde{i}\widetilde{n}} y_j - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)(j/n) - \sigma\epsilon_j + \widehat{g}^*(j/n) - \widehat{g}(j/n) \right| + 7C_k(t) \\ & \leq \left| \widetilde{n}^{-1} \sum_{j=(i-1)\widetilde{n}+1}^{\widetilde{i}\widetilde{n}} \left(g(j/n) - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)(j/n)\right) \right| + 8C_k(t) \\ & \leq |f(i/c)| + 8C_k(t) + \zeta, \end{aligned}$$

where

$$\zeta = \max_{1 \leq i \leq c} \left| f(i/c) - \frac{c}{n} \sum_{j=(i-1)\widetilde{n}+1}^{\widetilde{i}\widetilde{n}} \left(g(j/n) - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)(j/n)\right) \right| = O(n^{-1}).$$

Suppose that Condition **(B)** holds, and consider events

$$\begin{aligned}\mathcal{E}_1^* &= \{\sigma|\epsilon_i| + c_s\rho \leq \sqrt{\mathcal{T}_n} \text{ for all } 1 \leq i \leq n\}, \\ \mathcal{E}_2^* &= \{|g(i/n) - \widehat{g}(i/n)| \leq \sqrt{\mathcal{T}_n} \text{ for all } 1 \leq i \leq n\}, \\ \mathcal{E}_3^* &= \{|\mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)(i/n) - \widehat{g}^*(i/n)| \leq \sqrt{\mathcal{T}_n} \text{ for all } 1 \leq i \leq n\}.\end{aligned}$$

It is easy to show that $P(\mathcal{E}_1^* \cap \mathcal{E}_2^* \cap \mathcal{E}_3^*) \rightarrow 1$ as $n \rightarrow \infty$. Let $\mathcal{E}^* = \mathcal{E}_1^* \cap \mathcal{E}_2^* \cap \mathcal{E}_3^*$. Thus, we choose N'_η s.t. $P(\mathcal{E}^*) \geq 1 - \eta/3$ if $n \geq N'_\eta$. Define $\omega^{\text{linear}} = (\omega_1^{\text{linear}}, \dots, \omega_c^{\text{linear}})^T$, where $\omega_i^{\text{linear}} = \widetilde{z}_i^{\text{linear}} - \widetilde{z}_{i,0}^{\text{linear}}$. Obviously, $\omega_i^{\text{linear}} = z_i^{\text{linear}} - z_{i,0}^{\text{linear}}$ under event \mathcal{E}^* . Therefore, by (71), under event \mathcal{E}^* , we have $|\omega_i^{\text{linear}} - f(i/c)| \leq 8C_k(t) + \zeta$. Since $A \leq I_c$, we get that

$$(\omega^{\text{linear}} - \mathbf{f})^T A (\omega^{\text{linear}} - \mathbf{f}) \leq \sum_{i=1}^c (\omega_i^{\text{linear}} - f(i/c))^2 \leq 128cC_k(t)^2 + 2c\zeta^2, \quad (72)$$

where $\mathbf{f} = (f(1/c), \dots, f(c/c))^T$. By Lemma 16 and (72), we get the following lower bound:

$$\begin{aligned}(\omega^{\text{linear}})^T A \omega^{\text{linear}} &\geq \frac{1}{2} \mathbf{f}^T A \mathbf{f} - (\omega^{\text{linear}} - \mathbf{f})^T A (\omega^{\text{linear}} - \mathbf{f}) \\ &= \frac{c}{2} \|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c^2 - \frac{1}{2} \mathbf{f}^T (I_c - A) \mathbf{f} - (\omega^{\text{linear}} - \mathbf{f})^T A (\omega^{\text{linear}} - \mathbf{f}) \\ &\geq \frac{c}{2} \|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c^2 - \frac{1}{2} \varrho c (\lambda + c^{-2m}) \rho^2 - 128cC_k(t)^2 - 2c\zeta^2.\end{aligned} \quad (73)$$

Using a similar argument in Lemma 14, and the facts that $\text{Var}(z_i^{\text{linear}}) = \text{Var}(z_i^0)(1 + o_p(1)) = \tau_k^2(1 + o_p(1))$, $C_k(t)^2 \ll \tau_k^2$, as $n, c \rightarrow \infty$, one has that

$$\frac{E\{ |(\omega^{\text{linear}})^T A z_0^{\text{linear}}|^2 \}}{(1 + (ch^2)^{-1})\tau_k^2 \sum_{i=1}^c (|f(i/c)| + 8C_k(t) + \zeta)^2} \leq 8.$$

Therefore, there exists N'' s.t., when $c \geq N''$, $P(\mathcal{E}_2) \geq 1 - \eta/3$, where event \mathcal{E}_2 is defined as

$$\mathcal{E}_2 = \left\{ |(\omega^{\text{linear}})^T A z_0^{\text{linear}}| \leq C'_\eta \sqrt{1 + (ch^2)^{-1}} \tau_k \sqrt{\sum_{i=1}^c (|f(i/c)| + 8C_k(t) + \zeta)^2} \right\}, \quad (74)$$

and $C'_\eta = \sqrt{24/\eta}$.

From Theorem 6, it is straightforward to show that

$$\frac{(z_0^{\text{linear}})^T A z_0^{\text{linear}} - \text{trace}(A) \widehat{\tau}_k^2}{s_c \widehat{\tau}_k^2} = O_p(1).$$

Thus, there exists $C''_\eta > 0$ s.t. $P(\mathcal{E}_3) \geq 1 - \eta/3$ for all $c \geq N'_\eta$ and N'' , where

$$\mathcal{E}_3 = \left\{ \left| \frac{(z_0^{\text{linear}})^T A z_0^{\text{linear}} - \text{trace}(A) \widehat{\tau}_k^2}{s_c \widehat{\tau}_k^2} \right| \leq C''_\eta \right\}.$$

Then $P(\mathcal{E}^* \cap \mathcal{E}_2 \cap \mathcal{E}_3) \geq 1 - \eta$ for any $c \geq N'_\eta$ and N'' .

Suppose $g \in S_\rho^m(\mathbb{I})$ satisfies

$$\|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c = \sqrt{c^{-1} \sum_{i=1}^c \left[\frac{c}{2} \int_{\max(i/c-c^{-1}, 0)}^{\min(i/c+c^{-1}, 1)} [g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)](s) ds \right]^2} = \sqrt{\sum_{i=1}^c f^2(i/c)/c} > C_\eta \delta_{\text{linear}}^*,$$

where

$$C_\eta = \max \left\{ 6\varrho\rho^2, \sqrt{1536}, (72C'_\eta)^2, 6(C''_\eta + z_{1-\alpha/2} + 1) \right\}, \quad (75)$$

$$\delta_{\text{linear}}^* = \sqrt{c^{-1} \tau_{\text{linear}}^2 (1 + s_c + (ch^2)^{-1}) + \lambda + c^{-2m} + C_k(t)^2 + \zeta^2}.$$

Then, under event $\mathcal{E}^* \cap \mathcal{E}_2 \cap \mathcal{E}_3$, we have

$$\begin{aligned} & \frac{cT_{\text{linear}, \mu, t, c} - \text{trace}(A)\widehat{\tau}_k^2}{s_c \widehat{\tau}_k^2} \\ &= \frac{(z^{\text{linear}})^T A z^{\text{linear}} - (z_0^{\text{linear}})^T A z_0^{\text{linear}}}{s_c \widehat{\tau}_k^2} + \frac{(z_0^{\text{linear}})^T A z_0^{\text{linear}} - \text{trace}(A)\widehat{\tau}_k^2}{s_c \widehat{\tau}_k^2} \\ &\geq \frac{(z^{\text{linear}})^T A z^{\text{linear}} - (z_0^{\text{linear}})^T A z_0^{\text{linear}}}{s_c \widehat{\tau}_k^2} - C''_\eta \\ &= \frac{(\omega^{\text{linear}})^T A \omega^{\text{linear}} + 2(\omega^{\text{linear}})^T A z_0^{\text{linear}}}{s_c \widehat{\tau}_k^2} - C''_\eta \\ &\geq \frac{\frac{c}{2} \|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c^2 - \frac{1}{2} \varrho c (\lambda + c^{-2m}) \rho^2 - 128c C_k(t)^2 - 2c\zeta^2 - 2C'_\eta \tau_k \sqrt{(1 + \frac{1}{ch^2}) \sum_{i=1}^c (|f(i/c)| + 8C_k(t) + \zeta)^2}}{s_c \widehat{\tau}_k^2} - C''_\eta \\ &\geq \frac{\frac{c}{2} \|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c^2 - \frac{1}{2} \varrho c (\lambda + c^{-2m}) \rho^2 - 128c C_k(t)^2 - 2c\zeta^2 - 6C'_\eta \sqrt{1 + (ch^2)^{-1}} \tau_k \sqrt{c} \|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c}{s_c \widehat{\tau}_k^2} - C''_\eta \end{aligned} \quad (76)$$

$$\begin{aligned} &= \frac{\frac{c}{2} \|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c^2 \left(1 - \frac{\frac{1}{2} \varrho c (\lambda + c^{-2m}) \rho^2}{\frac{c}{2} \|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c^2} - \frac{128c C_k(t)^2}{\frac{c}{2} \|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c^2} - \frac{2c\zeta^2}{\frac{c}{2} \|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c^2} - \frac{6C'_\eta \sqrt{1 + (ch^2)^{-1}} \tau_k \sqrt{c} \|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c}{\frac{c}{2} \|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c^2} \right)}{s_c \widehat{\tau}_k^2} - C''_\eta \\ &\geq \frac{\frac{c}{6} \|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c^2}{s_c \widehat{\tau}_k^2} - C''_\eta > z_{1-\alpha/2}. \end{aligned} \quad (77)$$

where (76) follows from $C_\eta > 192$ (see (75)), i.e.,

$$\sqrt{\sum_{i=1}^c f(i/c)^2} = \sqrt{c} \|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c \geq \sqrt{c} C_\eta \delta_{\text{linear}}^* \geq \sqrt{192c} C_k(t),$$

$$\sqrt{\sum_{i=1}^c f(i/c)^2} = \sqrt{c} \|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c \geq \sqrt{c} C_\eta \delta_{\text{linear}}^* \geq \sqrt{3c} \zeta,$$

which leads to

$$\sum_{i=1}^c (|f(i/c)| + 8C_k(t) + \zeta)^2 \leq 3 \sum_{i=1}^c f(i/c)^2 + 192c C_k(t)^2 + 3c\zeta^2 \leq 9 \sum_{i=1}^c f(i/c)^2 = 9c \|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c^2;$$

and (77) follows from (75), i.e.,

$$\begin{aligned} \frac{\frac{1}{2}\varrho c(\lambda + c^{-2m})\rho^2}{\frac{c}{2}\|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c^2} &\leq 1/6, \\ \frac{128cC_k(t)^2}{\frac{c}{2}\|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c^2} &\leq 1/6, \\ \frac{2c\zeta^2}{\frac{c}{2}\|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c^2} &\leq 1/6, \\ \frac{6C'_\eta\sqrt{1 + (ch^2)^{-1}}\tau_k\sqrt{c}\|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c}{\frac{c}{2}\|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c^2} &\leq 1/6. \end{aligned}$$

Then for any $c \geq N_\eta \equiv \max\{N'_\eta, N''\}$, we have

$$P(\text{reject } H_0 | H_1 \text{ is true}) \geq P\left(\mathcal{E}^* \cap \mathcal{E}_2 \cap \mathcal{E}_3 \text{ and } \left| \frac{cT_{\text{linear}, \mu, t, c} - \text{trace}(A)\widehat{\tau}_k^2}{s_c\widehat{\tau}_k^2} \right| \geq z_{1-\alpha/2} \right) \geq 1 - \eta.$$

In the end, by direct calculations, we know that $\|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c \geq C_\eta \delta_{\text{linear}}^*$ is equivalent as $\|g - \mathcal{P}_{\mathcal{L}(\mathbb{I})}(g)\|_c \geq C_\eta \delta_{n,c,\lambda}^{\text{linear}}$. This proves the desired result. \blacksquare

Proof of Theorem 8: Let $\epsilon = (\epsilon_1, \dots, \epsilon_n)^T \sim N(0, I_n)$ and $\tilde{y}^0 = (\tilde{y}_1^0, \dots, \tilde{y}_c^0)^T$, where $\tilde{y}_i^0 = \frac{\sum_{j=(i-1)\tilde{n}+1}^{i\tilde{n}} \sigma \epsilon_j}{\tilde{n}}$ follows a normal distribution. We further define $\varsigma_i = z_i^0 - \tilde{y}_i^0$ be the difference of z_i^0 and \tilde{y}_i^0 . Let $\varsigma = (\varsigma_1, \dots, \varsigma_c)^T$. Consider event

$$\mathcal{E}_1 = \{\sigma|\epsilon_i| + c_s\rho \leq \sqrt{\mathcal{T}_n} \text{ for all } 1 \leq i \leq n\}.$$

Then $P(\mathcal{E}_1) \rightarrow 1$ as $n \rightarrow \infty$, and under event \mathcal{E}_1 , $|\tilde{y}_i^0 - \frac{1}{\tilde{n}} \sum_{j=(i-1)\tilde{n}+1}^{i\tilde{n}} Q(\sigma \epsilon_j)| \leq C_k(t)$. According to (33), one has that $|\varsigma_i| = O_p(C_k(t))$. Note that for any given $m_l \leq m \leq m_u \rightarrow \infty$, the standardized testing statistic

$$\begin{aligned} \xi_m &= \frac{cT_m - \text{trace}(A_m)\widehat{\tau}_k^2}{s_{c,m}\widehat{\tau}_k^2} \\ &= \frac{(\varsigma + \tilde{y}^0)^T A_m (\varsigma + \tilde{y}^0) - \text{trace}(A_m)\widehat{\tau}_k^2}{s_{c,m}\widehat{\tau}_k^2} \\ &= \frac{\varsigma^T A_m \varsigma}{s_{c,m}\widehat{\tau}_k^2} + \frac{2\varsigma^T A_m \tilde{y}^0}{s_{c,m}\widehat{\tau}_k^2} + \frac{(\tilde{y}^0)^T A_m \tilde{y}^0 - \text{trace}(A_m)\widehat{\tau}_k^2}{s_{c,m}\widehat{\tau}_k^2} \\ &= J_1 + J_2 + J_3. \end{aligned}$$

For J_1 , notice $\varsigma^T A_m \varsigma \leq \sum_{i=1}^c \varsigma_i^2 = O_p(cC_k(t)^2)$, and $C_k(t)^2 \ll (nh_m^{1/2})^{-1}$ for any $m_l \leq m \leq m_u$, one has $J_1 = o_p(1)$. For J_2 , using a similar argument as (60), we have

$$J_2 = O_p\left(C_k(t)\sqrt{\frac{n}{ch_m}} + C_k(t)\sqrt{nh_m}\right) \rightarrow 0, \quad \text{for any } m_l \leq m \leq m_u.$$

For J_3 , we need to use Lemma 17. Let $\tilde{A}_m = A_m/s_{c,m}$, $\Psi_c = \tilde{n}^{-1/2}(\tilde{y}_1^0, \dots, \tilde{y}_c^0)^T$. Define $F_{c,m} := \Psi_c^T \tilde{A}_m \Psi_c - E[\Psi_c^T \tilde{A}_m \Psi_c]$. Define $Z_c = (Z_{c,m_l}, \dots, Z_{c,m_u})^T$ be an $m_u - m_l + 1$

-dimensional centered Gaussian vector with covariance matrix $\mathfrak{C} = I_{m_u - m_l + 1}$. Next we need to verify the conditions in Lemma 17.

By direct calculations, we have $E[F_{c,m}^4] - 3E[F_{c,m}^2]^2 = 48\text{trace}(\tilde{A}_m^4) \asymp \frac{h_m}{s_{c,m}^4} \asymp h_m^3$. Then we have $\max_{m_l \leq m \leq m_u} \left(E[F_{c,m}^4] - 3E[F_{c,m}^2]^2 \right) \log^6 m_u \rightarrow 0$.

On the other hand,

$$\max_{m_l \leq m, m' \leq m_u} |\mathfrak{C}(m, m') - E[F_{c,m}F_{c,m'}]| \leq \sum_{m=m_l}^{m_u} |1 - E[F_{c,m}^2]| + 2 \sum_{m_l \leq m < m' \leq m_u} |E[F_{c,m}F_{c,m'}]|. \quad (78)$$

For the first term in (78), recall that $s_{c,m}^2 = 2 \sum_{1 \leq i \neq i' \leq c} a_{i,i'}^2$ with $a_{i,i'}$ being the (i, i') th entry of A_m . Then by Lemma 18, we have,

$$\sum_{m=m_l}^{m_u} |1 - E[F_{c,m}^2]| \log^2 m_u = \sum_{m=m_l}^{m_u} |1 - 2\text{trace}(\tilde{A}_m^2)| \log^2 m_u \asymp \frac{m_u \log^2(m_u)}{ch_m} \rightarrow 0. \quad (79)$$

For the second term in (78), we need to find a bound of $|E [F_{c,m}F_{c,m'}]|$ for $m' > m$. It follows that

$$\begin{aligned}
 E [F_{c,m}F_{c,m'}] &= 2\text{trace}(A_m A_{m'}) / (s_{c,m} s_{c,m'}) \\
 &\asymp \frac{1}{(h_m h_{m'})^{-1/2}} \sum_{\nu=1}^n \frac{\nu^{-4m}}{(\lambda_m + \nu^{-2m})^2} \frac{\nu^{-4m'}}{(\lambda_{m'} + \nu^{-2m'})^2} \\
 &= \frac{1}{(h_m h_{m'})^{-1/2}} \sum_{\nu=1}^n \frac{1}{(1 + \lambda_m \nu^{2m})^2 (1 + \lambda_{m'} \nu^{2m'})^2} \\
 &\leq \frac{1}{(h_m h_{m'})^{-1/2}} \sum_{\nu=1}^n \frac{1}{\left(1 + (h_m \nu)^{2m}\right) \left(1 + (h_{m'} \nu)^{2m'}\right)} \\
 &\leq \frac{1}{(h_m h_{m'})^{-1/2}} \sum_{\nu=1}^n \int_{\nu^{-1}}^{\nu} \frac{1}{\left(1 + (h_m x)^{2m}\right) \left(1 + (h_{m'} x)^{2m'}\right)} dx \\
 &= \frac{1}{(h_m h_{m'})^{-1/2}} \int_0^{\infty} \frac{1}{\left(1 + (h_m x)^{2m}\right) \left(1 + (h_{m'} x)^{2m'}\right)} dx \\
 &= \int_0^{\infty} \frac{\sqrt{h_m h_{m'}}}{\left(1 + (h_m x)^{2m}\right) \left(1 + (h_{m'} x)^{2m'}\right)} dx \\
 &= \int_0^{\infty} \frac{1}{\left(1 + \left(x \sqrt{h_m/h_{m'}}\right)^{2m}\right) \left(m' + \left(x \sqrt{h_{m'}/h_m}\right)^{2m'}\right)} dx \\
 &\leq \int_0^{\infty} \frac{1}{1 + \left(x \sqrt{h_{m'}/h_m}\right)^{2m'}} dx \\
 &= (h_{m'}/h_m)^{-1/2} \int_0^{\infty} \frac{1}{1 + x^{2m'}} dx \\
 &= (h_{m'}/h_m)^{-1/2} \left(\int_0^1 \frac{1}{1 + x^{2m'}} dx + \int_1^{\infty} \frac{1}{1 + x^{2m'}} dx \right) \\
 &\leq C_0 (h_{m'}/h_m)^{-1/2}.
 \end{aligned}$$

Therefore, using Lemma 18, we have

$$2 \sum_{m_l \leq m < m' \leq m_u} |E [F_{c,m}F_{c,m'}]| \log^2(m_u) \lesssim (h_{m'}/h_m)^{-1/2} m_u^2 \log^2(m_u) \rightarrow 0.$$

Together with (79), we have $\max_{m_l \leq m, m' \leq m_u} |\mathfrak{C}(m, m') - E [F_{c,m}F_{c,m'}]| \log^2 m_u \rightarrow 0$.

Therefore, by Lemma 17, we have

$$\sup_{x \in \mathbb{R}} \left| P \left(C_n \left(\max_{m_l \leq m_* \leq m_u} \xi_m - C_n \leq x \right) \right) - P \left(C_n \left(\max_{m_l \leq m \leq m_u} Z_{c,m} - C_n \leq x \right) \right) \right| \rightarrow 0, \text{ as } c \rightarrow \infty.$$

By Hall (1979), we know $C_n(\max_{m_l \leq m \leq m_u} Z_{c,m} - C_n)$ follows an extreme value distribution. Proof is complete. \blacksquare

Proof of Theorem 9: The proof of Theorem 9 is similar to Theorem 5 and Theorem 7. We use the same notations as in the proof of Theorem 5. Suppose $g \in S_\rho^{m_*}(\mathbb{I})$ is the function which generates the samples and f is the corresponding integral function as defined in (15). We consider the following three events as defined in the proof of Theorem 5.

$$\begin{aligned}\mathcal{E}_1 &= \{\sigma|\epsilon_i| + c_s\rho \leq \sqrt{\mathcal{T}_n} \text{ for all } 1 \leq i \leq n\}, \\ \mathcal{E}_2 &= \left\{ |\omega^T A_{m_*} z^0| \leq C'_\eta \sqrt{1 + (ch_{m_*}^2)^{-1} \tau_k} \sqrt{\sum_{i=1}^c (|f(i/c)| + 4C_k(t) + \zeta)^2} \right\}, \quad C'_\eta = \sqrt{24/\eta}, \\ \mathcal{E}_3 &= \left\{ \left| \frac{(z^0)^T A_{m_*} z^0 - \text{trace}(A_{m_*}) \widehat{\tau}_k^2}{s_{c,m_*} \widehat{\tau}_k^2} \right| \leq C''_\eta \right\}.\end{aligned}$$

Since $P(\mathcal{E}_1) \rightarrow 1$ as $c \rightarrow \infty$, there exist $N'_\eta > 0$, such that $P(\mathcal{E}_1) \geq 1 - \eta/3$ for all $c \geq N'_\eta$. Follows from Lemma 14 that there exists N'' s.t., when $c \geq N''$, $P(\mathcal{E}_2) \geq 1 - \eta/3$. Furthermore, using Theorem 3, there exists $C''_\eta > 0$ s.t. $P(\mathcal{E}_3) \geq 1 - \eta/3$ for all $c \geq \max\{N'_\eta, N''\}$.

Suppose $g \in S_\rho^{m_*}(\mathbb{I})$ satisfies $\|g\|_c \geq C_\eta \delta_*$, where

$$\begin{aligned}C_\eta &= \max\{6\varrho\rho^2, 384, (72C'_\eta)^2, 6(C''_\eta + z_{1-\alpha/2} + 1)\}, \\ \delta_* &= \sqrt{c^{-1}\tau_k^2(1 + s_{c,m_*} \log^{1/2}(m_u) + (ch_{m_*}^2)^{-1}) + \lambda + c^{-2m_*} + C_k(t)^2 + \zeta^2}, \\ \tau_k^2 &= \text{Var}(z_1|H_0) = \text{Var}(z_i^0) = O(\tilde{n}^{-1}), \quad \zeta = \max_{i=1,\dots,c} \left| f(i/c) - \frac{1}{\tilde{n}} \sum_{j=(i-1)\tilde{n}+1}^{\tilde{n}} g(j/\tilde{n}) \right| = O(n^{-1}).\end{aligned}$$

Since $m_u \rightarrow \infty, m_* \leq m_u$ eventually. So we assume $m_* \leq m_u$. Then it holds that

$$\begin{aligned}\inf_{\substack{g \in S_\rho^{m_*}(\mathbb{I}) \\ \|g\|_c \geq C_\eta \delta_*}} P(\xi_* > q_\alpha) &= \inf_{\substack{g \in S_\rho^{m_*}(\mathbb{I}) \\ \|g\|_c \geq C_\eta \delta_*}} P(\xi_{\max} > C_n + q_\alpha/C_n) \\ &\geq \inf_{\substack{g \in S_\rho^{m_*}(\mathbb{I}) \\ \|g\|_c \geq C_\eta \delta_*}} P(\xi_{m_*} > C_n + q_\alpha/C_n) \\ &= \inf_{\substack{g \in S_\rho^{m_*}(\mathbb{I}) \\ \|g\|_c \geq C_\eta \delta_*}} P\left(\frac{cT_{m_*} - \text{trace}(A_{m_*}) \widehat{\tau}_k^2}{s_{c,m_*} \widehat{\tau}_k^2} > C_n + q_\alpha/C_n\right).\end{aligned}$$

Similar to the proof of Theorem 5, we know with probability approaching one

$$\begin{aligned}
 & \frac{cT_{m_*} - \text{trace}(A_{m_*})\hat{\tau}_k^2}{s_{c,m_*}\hat{\tau}_k^2} \\
 = & \frac{z^T A_{m_*} z - (z^0)^T A_{m_*} z^0}{s_{c,m_*}\hat{\tau}_k^2} + \frac{(z^0)^T A_{m_*} z^0 - \text{trace}(A_{m_*})\hat{\tau}_k^2}{s_{c,m_*}\hat{\tau}_k^2} \\
 = & \frac{z^T A_{m_*} z - (z^0)^T A_{m_*} z^0}{s_{c,m_*}\hat{\tau}_k^2} + O_p(1) \\
 = & \frac{\omega^T A_{m_*} \omega + 2\omega^T A_{m_*} z^0}{s_c \hat{\tau}_k^2} + O_P(1) \\
 \geq & \frac{\frac{c}{2}\|g\|_c^2 \left(1 - \frac{\frac{1}{2}c_{m_*}c(\lambda_{m_*} + c^{-2m_*})\rho^2}{\frac{c}{2}\|g\|_c^2} - \frac{32cC_k(t)^2}{\frac{c}{2}\|g\|_c^2} - \frac{2c\zeta^2}{\frac{c}{2}\|g\|_c^2} - \frac{6C'_\eta\sqrt{1+(ch_{m_*}^2)^{-1}\tau_k\sqrt{c}\|g\|_c}}{\frac{c}{2}\|g\|_c^2} \right)}{s_{c,m_*}\hat{\tau}_k^2} + O_p(1).
 \end{aligned}$$

Since $C_n \asymp (\log(m_u))^{1/2}$ we have

$$\frac{cT_{m_*} - \text{trace}(A_{m_*})\hat{\tau}_k^2}{s_{c,m_*}\hat{\tau}_k^2} \geq C_n + q_\alpha/C_n.$$

Therefore, for any $c \geq N_\eta \equiv \max\{N'_\eta, N''\}$, we have

$$P(\text{reject } H_0 | H_1 \text{ is true}) \geq P\left(\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3 \text{ and } \left| \frac{cT_{m_*} - \text{trace}(A_{m_*})\hat{\tau}_k^2}{s_{c,m_*}\hat{\tau}_k^2} \right| \geq C_n + q_\alpha/C_n\right) \geq 1 - \eta.$$

In the end, by direct calculations, we know that $\|g\|_c \geq C_\eta \delta_*$ is equivalent as $\|g\|_c \geq C_\eta \delta_{n,c,a_n}$. This proves the desired result. \blacksquare