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Towards Unified Native Spaces in Kernel Methods

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Abstract

There exists a plethora of parametric models for positive definite kernels in Euclidean spaces, and their use is ubiquitous in statistics, machine learning, numerical analysis, and approximation theory. Usually, the kernel parameters index certain features of an associated process. Amongst those features, smoothness (in the sense of Sobolev spaces, mean square differentiability, and fractal dimensions), compact or global supports, and negative dependencies (hole effects) are of interest to several theoretical and applied disciplines. This paper unifies a wealth of well-known kernels into a single parametric class that encompasses them as special cases, attained either by exact parameterization or through parametric asymptotics. We furthermore find parametric restrictions under which we can characterize the Sobolev space that is norm equivalent to the RKHS associated with the new kernel. As a by-product, we infer the Sobolev spaces that are associated with existing classes of kernels. We illustrate the main properties of the new class, show how this class can switch from compact to global supports, and provide special cases for which the kernel attains negative values over nontrivial intervals. Hence, the proposed class of kernel is the reproducing kernel of a Hilbert space that contains many special cases, including the celebrated Matérn and Wendland kernels, as well as their aliases with hole effects.

Keywords: RKHS, Sobolev spaces, smoothness, hole effects, compactly supported kernels

1. Introduction

The terminology *Native Spaces* for reproducing kernel Hilbert spaces (RKHSs) associated with given classes of positive definite kernels was introduced by Schaback (1995). Positive definite kernels and native spaces are now used in approximation theory, numerical analysis, computational science, signal processing, machine learning, statistics and probability, with a monumental literature from all these disciplines and with countless applications in science and engineering. In this paper, one of the most important aspects is provided by the connection between certain parametric classes of kernels and their native spaces that are norm equivalent to certain classes of Sobolev spaces. This aspect puts *smoothness* into play, and smoothness plays a dominant role in the aforementioned disciplines, among others.

1.1 Features of Interest

Smoothness. The local behavior of a stationary Gaussian random field in the d-dimensional Euclidean space exclusively depends on its covariance kernel; in particular, the sample paths are k-times differentiable in the mean-square sense if, and only if, the kernel is $2k$-times differentiable at the origin. Under the same condition, the sample paths have (local) Sobolev space exponent being identically equal to k.

Support. Compactly supported kernels are relevant in several disciplines. To mention:

1. Compact support implies sparse covariance matrices, which considerably reduces the computational burden to solve linear systems of equations needed in spatial statistics for maximum likelihood estimations or for prediction by kriging (Furrer et al., 2006).

2. The discrete Fourier spectra of the covariance matrix computed on a sufficiently large spatial domain has nonnegative entries, which allows the exact simulation of Gaussian random fields on regular grids (Chilès and Delfiner, 2012, Section 7.5.4). Without the compact support restriction, the discrete spectral simulation becomes approximate and does not accurately reproduce the spatial correlation structure of the target Gaussian random field.

3. An isotropic kernel on the d-dimensional unit sphere can be constructed from an isotropic kernel in the d-dimensional Euclidean space by substituting the Euclidean distance by the geodesic (great-circle) distance on the sphere, provided that d is odd and that the kernel is supported in $[0, \pi]$ (Emery et al., 2023).

4. Transitive covariograms and geometric covariograms are compactly supported kernels used in geostatistics, mathematical morphology, and stochastic geometry (Matheron, 1965; Serra, 1982).

Hole effects. Kernels attaining negative values are said to have a hole effect (Chilès and Delfiner, 2012), a feature of interest in applications to the natural sciences and engineering.

For example, it can reveal sedimentary processes in geology, competition processes in ecology, or anthropogenic processes in agronomy (Alegría and Emery, 2024, Supplementary material). Another example of hole effect arises in time series analysis (Hurd and Miamee, 2007) or image analysis (Bonetto et al., 2002) when data exhibit a quasi-periodic behavior.

1.2 Challenges and Contribution

The aforementioned features (smoothness, compact support, and hole effect) have hardly been considered in a single class of positive definite kernels. Substantially, the literature has challenged the smoothness problem using the Matérn class, and the computational problem using the compactly supported Generalized Wendland class. As a result, the literature is fragmented, with a clear lack of connections. Additionally, many radially symmetric kernels proposed in earlier literature have not been properly characterized in terms of smoothness.

This paper presents a new class of radially symmetric positive definite kernels that includes most parametric classes of kernels from previous literature as special or asymptotic cases. Importantly, it is worth emphasizing that this unified class remains parsimonious, as it is indexed by only 5 scalar parameters that relate to the smoothness, support (range), hole effect, behavior near the range and shape of the kernel, that is, much fewer parameters than naive constructions based on convex linear combinations of known kernels.

We furthermore find the Sobolev space that is norm equivalent to the RKHS associated with this new class. As a by-product, most of the well-known classes of positive definite kernels are implicitly characterized in terms of smoothness. We also study the local properties of the proposed kernel, which in turn determines the mean square differentiability of associated Gaussian random fields and their fractal dimensions. We provide a characterization of these properties for the majority of kernels proposed in earlier literature.

1.3 How to Read this Paper

Readers unfamiliar with mathematics may look at Table 2 and Figure 1 next to appraise the impact of this paper. A wealth of well-known kernels are included as special or asymptotic cases of the new kernel and implicitly characterized in terms of smoothness. Practitioners can now control the three main modeling features—smoothness, compact support and hole effect—through a single class of kernels.

The outline is as follows. A background in Section 2 illustrates the connections between positive definite kernels, RKHSs and Sobolev spaces. Section 3 presents the new class of kernels and the parametric space for which the kernel is *permissible* (read: positive definite). Further, we characterize the spectral density related to this class. An asymptotic argument will allow to determine the Sobolev space that is norm equivalent to the native space associated with the new class. This section also provides a study of the local properties of Gaussian random fields with the new covariance kernels proposed here. The special cases indicated in Table 2 are attained through specific parameterizations or parametric asymptotics (Sections 4 and 5). Sections 6 and 7 illustrate the consequences and relevance of our findings for the fields of statistics and machine learning. Concluding remarks are provided in Section 8. Readers interested in the mathematical proofs can further integrate the reading through Appendices A to C, which contain many results of independent interest.

2. Background

The notation and special functions indicated in Table 1 will be used in this paper.

i	Complex imaginary unit
$\mathbb{R}_{>\alpha}$	Set of real numbers greater than α
$\mathbb{N}_{\geq\alpha}$	Set of integers greater than, or equal to, α ($\mathbb{N} = \mathbb{N}_{\geq 0}$)
$\langle\cdot,\cdot\rangle_d$	Inner product in $\mathbb{R}^d$
$\Vert\cdot\Vert_d$	Euclidean norm in $\mathbb{R}^d$
$(\cdot)_+$	Positive part function
$\lfloor\cdot\rfloor$	Floor function
$\lceil\cdot\rceil$	Ceil function
$(\cdot)_n$	Pochhammer symbol
Γ	Gamma function
$\Gamma^+(\cdot,\cdot)$	Upper incomplete Gamma function
$\Gamma^-(\cdot,\cdot)$	Lower incomplete Gamma function
J_ν	Bessel function of the first kind
K_ν	Modified Bessel function of the second kind
L_n^μ	Generalized Laguerre polynomial
${}_2F_1(\begin{smallmatrix}\alpha,\beta\\ \gamma\end{smallmatrix};\cdot)$	Gauss hypergeometric function, with α,β,γ real
${}_pF_q(\begin{smallmatrix}\boldsymbol{\beta}\\ \boldsymbol{\gamma}\end{smallmatrix};\cdot)$	Generalized hypergeometric function, with $p,q\in\mathbb{N}$, $\boldsymbol{\beta}\in\mathbb{R}^p$, $\boldsymbol{\gamma}\in\mathbb{R}^q$
$G_{p,q}^{m,n}(\cdot\vert\begin{smallmatrix}\boldsymbol{\beta}\\ \boldsymbol{\gamma}\end{smallmatrix})$	Meijer G-function, with $m,n,p,q\in\mathbb{N}$, $\boldsymbol{\beta}\in\mathbb{R}^p$, $\boldsymbol{\gamma}\in\mathbb{R}^q$

Table 1: Notation and special functions used in this paper (Olver et al., 2010)

2.1 Gaussian Random Fields, Kernels and Native Spaces

Let $d\in\mathbb{N}_{\geq 1}$ and $Z=\{Z(\boldsymbol{x}):\boldsymbol{x}\in\mathbb{R}^d\}$ be a zero-mean Gaussian random field having kernel (*covariance*) $K:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ defined through $K(\boldsymbol{x},\boldsymbol{x}'):=\mathrm{Cov}(Z(\boldsymbol{x}),Z(\boldsymbol{x}'))$. Covariance functions are symmetric and positive (semi)definite, that is $\sum_{i=1}^n\sum_{j=1}^n c_iK(\boldsymbol{x}_i,\boldsymbol{x}_j)c_j\geq 0$ for all $n\in\mathbb{N}$, $c_1,\ldots,c_n\in\mathbb{R}$ and $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\in\mathbb{R}^d$. If the above inequality is strict for $(c_1,\ldots,c_n)$ being non-zero, then K is called strictly positive definite. Positive definite (and symmetric) functions $K:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ determine *translate* functions $K(\boldsymbol{x},\cdot)$ on $\mathbb{R}^d$, for all $\boldsymbol{x}\in\mathbb{R}^d$. We define the inner product applying on pairs of translates through

$$\Big\langle K(\boldsymbol{x},\cdot),K(\boldsymbol{x}',\cdot)\Big\rangle_{\mathsf{H}(K)}:=K(\boldsymbol{x},\boldsymbol{x}'),\quad \boldsymbol{x},\,\boldsymbol{x}'\in\mathbb{R}^d. \tag{1}$$

Such an inner product extends to all linear combinations of translates and *generates*, by completion, a Hilbert space $\mathsf{H}(K)$ of functions on $\mathbb{R}^d$ called the *native* space for K (Schaback, 1995). Most often, this Hilbert space is a subspace of the space $L_2(\mathbb{R}^d)$ of continuous and square integrable functions in $\mathbb{R}^d$. As explained in Schaback (1995), the Hilbert space allows for continuous point evaluations $\delta_{\boldsymbol{x}} : f \mapsto f(\boldsymbol{x})$ via a *reproduction formula*

$$f(\boldsymbol{x}) = \Big\langle f, K(\boldsymbol{x}, \cdot) \Big\rangle_{\mathsf{H}(K)}, \qquad \boldsymbol{x} \in \mathbb{R}^d, \quad f \in \mathsf{H}(K),$$

which directly follows from (1). The native space $\mathsf{H}(K)$ is also called a *reproducing kernel Hilbert space* with reproducing kernel K. We note that the translates $K(\boldsymbol{x}, \cdot)$ lie in $\mathsf{H}(K)$, forming its completion and being the Riesz representers of delta functionals $\delta_{\boldsymbol{x}}$. Translates cover a central role in numerical analysis, approximation theory and machine learning, because the so-called kernel trick allows for computing inner products over the abstract space $\mathsf{H}(K)$.

Our paper deals with continuous and stationary kernels, that is $K(\boldsymbol{x}, \boldsymbol{x}') \equiv \mathsf{K}(\boldsymbol{x} - \boldsymbol{x}')$, such that K is absolutely integrable. The following Fourier identities hold (Yaglom, 1987):

$$\mathsf{K}(\boldsymbol{h}) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{h}\, , \boldsymbol{\omega} \rangle_d}\, \hat{\mathsf{K}}(\boldsymbol{\omega}) \mathrm{d}\boldsymbol{\omega}, \qquad \boldsymbol{h} \in \mathbb{R}^d,$$

$$\hat{\mathsf{K}}(\boldsymbol{\omega}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\langle \boldsymbol{h}, \boldsymbol{\omega} \rangle_d}\, \mathsf{K}(\boldsymbol{h}) \mathrm{d}\boldsymbol{h}, \qquad \boldsymbol{\omega} \in \mathbb{R}^d, \tag{2}$$

where $\hat{\mathsf{K}}$ is called the *spectral density* of K. For any function f in $\mathsf{H}(\mathsf{K})$, the Fourier transform $\hat{f}$ is defined as in (2). Fourier transforms can be used to recover the inner product (1) on the Hilbert space $\mathsf{H}(\mathsf{K})$ through

$$\langle f, g \rangle_{\mathsf{H}(\mathsf{K})} = \int_{\mathbb{R}^d} \frac{\hat{f}(\boldsymbol{\omega}) \overline{\hat{g}}(\boldsymbol{\omega})}{\hat{\mathsf{K}}(\boldsymbol{\omega})} \mathrm{d}\boldsymbol{\omega}, \qquad f, g \in \mathsf{H}(\mathsf{K}), \tag{3}$$

up to a constant factor (Porcu et al., 2024). Here, $\overline{\hat{g}}$ is the complex conjugate of $\hat{g}$. We can rephrase the above by saying that the space $\mathsf{H}(\mathsf{K})$ contains those functions f such that the ratio $\hat{f}\, \hat{\mathsf{K}}^{-1/2}$ is square integrable over $\mathbb{R}^d$. This creates a link with certain function spaces.

2.2 Spectral Representations of Stationary Isotropic Kernels

We now turn into the additional assumption of isotropy or radial symmetry, for which $K(\boldsymbol{x}, \boldsymbol{x}')$ exists for any $\boldsymbol{x}$ and $\boldsymbol{x}'$ in $\mathbb{R}^d$ and only depends on the distance $\|\boldsymbol{x} - \boldsymbol{x}'\|_d$:

$$K(\boldsymbol{x}, \boldsymbol{x}') = C\left(\|\boldsymbol{x} - \boldsymbol{x}'\|_d\right), \quad \boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R}^d. \tag{4}$$

Hereinafter, we denote Φ_d the class of continuous mappings $C : [0, +\infty) \to \mathbb{R}$ such that (4) is true for a second-order stationary isotropic Gaussian random field in $\mathbb{R}^d$. One has

$$\Phi_1 \supset \Phi_2 \supset \ldots \supset \Phi_d \supset \ldots \supset \Phi_\infty := \bigcap_{d=1}^{+\infty} \Phi_d.$$

Any member of Φ_d has the following representation (Schoenberg, 1938, Theorem 1):

$$C(h) = \int_0^{+\infty} \mathcal{J}_{1/u,d}(h) \mathrm{d}G_d(u), \qquad h \geq 0, \tag{5}$$

where G_d is a nondecreasing bounded measure on $(0, +\infty)$ (called Schoenberg measure by Daley and Porcu (2014)) and $\mathcal{J}_{a,d}$ is the Schoenberg (aka Bessel-J) kernel:

$$\mathcal{J}_{a,d}(h) := \begin{cases} \Gamma\left(\frac{d}{2}\right)\left(\frac{h}{2a}\right)^{1-\frac{d}{2}} J_{\frac{d}{2}-1}\left(\frac{h}{a}\right) \text{ if } h > 0 \\ 1 \text{ if } h = 0. \end{cases} \tag{6}$$

If, furthermore, $C(\|\cdot\|_d)$ is absolutely integrable in $\mathbb{R}^d$, then G_d is absolutely continuous with respect to the Lebesgue measure, that is, it has a density g_d such that

$$C(h) = 2^{\frac{d}{2}-1}\Gamma\left(\frac{d}{2}\right) h^{1-\frac{d}{2}} \int_0^{\infty} u^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(uh) g_d(u) \mathrm{d}u, \qquad h > 0, \tag{7}$$

where g_d is a nonnegative and integrable function on $[0, +\infty)$ that will be referred to as the d-radial Schoenberg density of C. On the other hand, for any member C of Φ_d such that $C(\|\cdot\|_d)$ is absolutely integrable, the following Fourier-Hankel representations hold:

$$\begin{aligned} C(h) &= (2\pi)^{\frac{d}{2}} h^{1-\frac{d}{2}} \int_0^{+\infty} u^{\frac{d}{2}} J_{\frac{d}{2}-1}(uh)\, \widehat{C}_d(u) \mathrm{d}u, \qquad h > 0, \\ \widehat{C}_d(u) &= \frac{1}{(2\pi)^{\frac{d}{2}}} u^{1-\frac{d}{2}} \int_0^{+\infty} h^{\frac{d}{2}} J_{\frac{d}{2}-1}(uh)\, C(h) \mathrm{d}h, \qquad u > 0, \end{aligned} \tag{8}$$

where $\widehat{C}_d : (0, +\infty) \to [0, +\infty)$ is the radial part of the spectral density $\hat{\mathsf{K}}$ of K, as per (2), and will be referred to as the d-radial spectral density of C. Comparing (7) and (8) gives

$$g_d(u) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} u^{d-1}\, \widehat{C}_d(u), \quad u > 0. \tag{9}$$

2.3 Sobolev Spaces

Consider the classical Sobolev space $H^s(\mathbb{R}^d) = \{f \in L_2(\mathbb{R}^d) : \hat{f}(\cdot)\,(1 + \|\cdot\|_d)^{\frac{s}{2}} \in L_2(\mathbb{R}^d)\}$ equipped with the inner product

$$\langle f, g \rangle_{H^s(\mathbb{R}^d)} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{\hat{f}(\boldsymbol{\omega})\overline{\hat{g}}(\boldsymbol{\omega})}{\left(1 + \|\boldsymbol{\omega}\|_d^2\right)^{-s}} \mathrm{d}\boldsymbol{\omega}. \tag{10}$$

This is identical to the inner product (3) under the special case $\hat{\mathsf{K}}(\boldsymbol{\omega}) = \left(1 + \|\boldsymbol{\omega}\|_d^2\right)^{-s}$. When $s = \nu + \frac{d}{2}$ with $\nu > 0$, this inner product corresponds precisely to the Matérn kernel $\mathsf{K}(\boldsymbol{h}) = \mathcal{M}_{1,\nu,d}(\|\boldsymbol{h}\|_d)$ (Porcu et al., 2024), with

$$\mathcal{M}_{a,\nu,d}(h) = \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\frac{h}{a}\right)^{\nu} K_\nu\left(\frac{h}{a}\right), \qquad h \geq 0, \quad a > 0, \quad \nu > 0. \tag{11}$$

Although the expression of the kernel (11) does not depend on d, we use it in the indices of parameters to emphasize the dimension of the Euclidean space under consideration. The kernel (11) actually belongs to Φ_∞, which is a subset of Φ_d.

By the Sobolev embedding theorem, the function space $H^s(\mathbb{R}^d)$ is contained in the space of continuous functions in $\mathbb{R}^d$. Arguments in Wendland (1995) in concert with a straight comparison between (3) and (10) provide the desired connection: if a kernel $C \in \Phi_d$ has a d-radial spectral density $\widehat{C}_d$ such that there exists constants $0 < c_1 < c_2 < +\infty$ with

$$c_1(1+u^2)^{-s} \le \widehat{C}_d(u) \le c_2(1+u)^{-s}, \qquad u \in (0,+\infty), \qquad s > \frac{d}{2}, \tag{12}$$

then the reproducing kernel associated with C is norm equivalent to the Sobolev space $H^s(\mathbb{R}^d)$. This is one of the reasons why the Matérn kernel has been so popular in statistics, machine learning, and numerical analysis.

For a Gaussian random field Z in $\mathbb{R}^d$, we consider mean square differentiability in the classical sense and we adopt the traditional definition of fractal dimension as Hausdorff dimension (Falconer, 2014). In particular, if the kernel $C \in \Phi_d$ is such that $(1 - C(h))h^{-\alpha}$ tends to 1 as h tends to 0 for some $\alpha \in (0, 2)$, then the fractal dimension of Z is $D = d+1-\frac{\alpha}{2}$ with probability 1 (Adler, 1981). Accordingly, both properties (differentiability and fractal dimension) are, for the case of Gaussian random fields, in one-to-one correspondence with the local properties of the associated covariance kernel.

3. The Class $\mathcal{H}$ of Generalized Hypergeometric Kernels

The following details the parametric family of kernels that motivates this paper.

Theorem 1 (Generalized hypergeometric kernel) *Let $a, \alpha, \beta, \gamma \in \mathbb{R}_{>0}$, $d \in \mathbb{N}_{\ge 1}$ and $k \in \mathbb{N}$. Let $\boldsymbol{\theta} = (a, \alpha, \beta, \gamma, d, k)$. The mapping $\mathcal{H}_{\boldsymbol{\theta}} : [0, +\infty) \to \mathbb{R}$, defined by*

$$\mathcal{H}_{\boldsymbol{\theta}}(h) = \varpi \left(\frac{h}{a}\right)^{2\alpha-d-2k} {}_3F_2\left(\begin{matrix}\alpha,\ 1+\alpha-\beta,\ 1+\alpha-\gamma \\ 1+\alpha-\frac{d}{2}-k,\ \alpha-k\end{matrix}; \frac{h^2}{a^2}\right) + {}_3F_2\left(\begin{matrix}\frac{d}{2}+k,\ 1+\frac{d}{2}+k-\beta,\ 1+\frac{d}{2}+k-\gamma \\ 1+\frac{d}{2}+k-\alpha,\ \frac{d}{2}\end{matrix}; \frac{h^2}{a^2}\right), \tag{13}$$

for $0 \le h < a$, and 0 otherwise, with $\varpi = \frac{\Gamma(\alpha)\Gamma(\beta-\frac{d}{2}-k)\Gamma(\gamma-\frac{d}{2}-k)\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}+k-\alpha)}{\Gamma(\frac{d}{2}+k)\Gamma(\alpha-\frac{d}{2}-k)\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)\Gamma(\alpha-k)}$, belongs to Φ_d provided the following parametric restrictions are adopted:

(A.1) $\alpha > \frac{d}{2} + k$;

(A.2) $2(\beta - \alpha)(\gamma - \alpha) \ge \alpha$;

(A.3) $2(\beta + \gamma) \ge 6\alpha + 1$;

(A.4) $\alpha - \frac{d}{2} - k \notin \mathbb{N}$. ■

Apart from the dimension d of the space $\mathbb{R}^d$ where $\mathcal{H}_{\boldsymbol{\theta}}(\|\cdot\|_d)$ is positive definite, the class $\mathcal{H}_{\boldsymbol{\theta}}$ has 5 parameters, and their role will be progressively illustrated below. Note that conditions (A.2) and (A.3) imply that both β and γ are greater than α.

Theorem 2 *Let $\mathcal{H}_{\boldsymbol{\theta}}$ be the kernel defined through (13) and let conditions (A.1) to (A.4) in Theorem 1 hold. Then, the function $\mathcal{H}_{\boldsymbol{\theta}}(\|\cdot\|_d)$ is absolutely integrable in $\mathbb{R}^d$ and possesses a uniquely determined d-radial spectral density, denoted $\widehat{\mathcal{H}}_{\boldsymbol{\theta}}$, admitting expression*

$$\widehat{\mathcal{H}}_{\boldsymbol{\theta}}(u) = \widehat{\varpi}\, a^{d+2k}\, u^{2k}\, {}_1F_2\left(\begin{smallmatrix} \alpha \\ \beta,\ \gamma \end{smallmatrix}; -\frac{a^2u^2}{4}\right), \tag{14}$$

for $u \in (0, +\infty)$, with $\widehat{\varpi} = \frac{\Gamma(\frac{d}{2})\Gamma(\alpha)\Gamma(\beta-\frac{d}{2}-k)\Gamma(\gamma-\frac{d}{2}-k)}{\pi^{\frac{d}{2}}2^{d+2k}\Gamma(\frac{d}{2}+k)\Gamma(\alpha-\frac{d}{2}-k)\Gamma(\beta)\Gamma(\gamma)}$. ∎

Theorem 3 *Let $\mathcal{H}_{\boldsymbol{\theta}}(\|\cdot\|_d)$ be the kernel defined through (13) and let conditions (A.1) to (A.4) in Theorem 1 hold. If, additionally, the inequality in (A.3) is strict, then $\mathcal{H}_{\boldsymbol{\theta}}(\|\cdot\|_d)$ is a reproducing kernel with RKHS that is norm equivalent to the Sobolev space $H^{\alpha-k}(\mathbb{R}^d)$.* ∎

This result, in concert with the findings proved in Appendix C, parameterizes the Sobolev spaces associated with most of the parametric classes of continuous correlation functions that are used in applications. Table 2 is a resumé of these kernels, being all of them members of Φ_d (with, possibly, restrictions on d as indicated in the table) and special cases or asymptotic cases of the class $\mathcal{H}_{\boldsymbol{\theta}}$. To understand the table, we split the cases into two classes: when $k = 0$, the class $\mathcal{H}_{\boldsymbol{\theta}}$ reduces to the Gauss Hypergeometric kernel introduced by Emery and Alegría (2022) (see Proposition 1 next). Hence, we report all the special cases according to either $k = 0$ or $k \neq 0$. In turn, for every class, we report (third column) the parametric restriction on $\mathcal{H}_{\boldsymbol{\theta}}$ that allows to attain the corresponding kernel as a special or asymptotic case. The fourth column allows to understand whether the specific result is being shown in this paper, or has been established by other authors. To provide further insight on this table, a graphical representation of the same in the form of diagramatic relation is reported in Figure 1.

Despite this versatility, our class does obviously not contain *all* the kernels proposed in earlier literature. In particular, the upgraded Euclid's hats, truncated polynomials, Askey and original Wendland kernels that belong to the $\mathcal{H}$ class partially overlap with Wu's (Wu, 1995) and Buhmann's (Buhmann, 2001) compactly supported kernels, but not all of the latter kernels are members of our generalized hypergeometric class. Also, kernels with a heavy tail describing random fields with a long memory, such as the inverse multiquadric (Schölkopf and Smola, 2002), generalized Cauchy (Gneiting and Schlather, 2004), Cauchy-Matérn (Alegría et al., 2024), or confluent hypergeometric (Yarger and Bhadra, 2025) kernels, are not covered by our class.

Model	Submodel	Restrictions	Reference
	Euclid's hat	$\alpha = \frac{d+1}{2}$, $\beta = \alpha + \frac{1}{2}$, $\gamma = 2\alpha$	Matheron (1965); Schaback (1995)
	Triangular	$\alpha = 1$, $\beta = \frac{3}{2}$, $\gamma = 2$, $d = 1$	Matérn (1960)
	Circular	$\alpha = \frac{3}{2}$, $\beta = 2$, $\gamma = 3$, $d = 2$	Matérn (1960)
	Spherical	$\alpha = 2$, $\beta = \frac{5}{2}$, $\gamma = 4$, $d = 3$	Matérn (1960)
	Pentaspherical	$\alpha = 3$, $\beta = \frac{7}{2}$, $\gamma = 6$, $d = 5$	Matérn (1960)
	Upgraded Euclid's hat	$\alpha > \frac{d}{2}$, $\beta = \alpha + \frac{1}{2}$, $\gamma = 2\alpha$	Matheron (1965)
	Cubic	$\alpha = 3$, $\beta = \frac{7}{2}$, $\gamma = 6$, $d = 3$	Chilès (1977)
	Penta	$\alpha = 4$, $\beta = \frac{9}{2}$, $\gamma = 8$, $d = 3$	Chilès and Delfiner (2012)
$\mathcal{H}$ class	Generalized Wendland	$\beta - \alpha \geq \frac{\alpha}{2} > \frac{d}{4}$, $\gamma = \beta + \frac{1}{2}$	Gneiting (2002); Zastavnyi (2006)
	Ordinary Wendland	$\alpha - \frac{d+1}{2} \in \mathbb{N}$, $\beta - \alpha \geq \frac{\alpha}{2}$, $\gamma = \beta + \frac{1}{2}$	Gneiting (1999a)
	Original Wendland	$\alpha - \frac{d+1}{2} \in \mathbb{N}$, $2(\beta - \alpha) \in \mathbb{N}_{\geq \alpha}$, $\gamma = \beta + \frac{1}{2}$	Wendland (1995)
($k = 0$)	Missing Wendland	$\alpha - \frac{d}{2} \in \mathbb{N}_{\geq 1}$, $2(\beta - \alpha) \in \mathbb{N}_{\geq \alpha}$, $\gamma = \beta + \frac{1}{2}$	Schaback (2011)
(Proposition 1)	Askey	$\alpha = \frac{d+1}{2}$, $\beta - \alpha \geq \frac{\alpha}{2}$, $\gamma = \beta + \frac{1}{2}$	Golubov (1981)
	Quadratic	$\alpha = 2, \beta = 3$, $\gamma = \frac{7}{2}$, $d = 3$	Alfaro (1984)
	Truncated power	$\alpha - \frac{d}{2} \in \mathbb{R}_{>0} \setminus \mathbb{N}$, $\beta - \alpha \in \mathbb{N}_{\geq 1}$, $\gamma - \frac{d}{2} \in \mathbb{N}_{\geq 1}$	Emery and Alegría (2022)
	Truncated polynomial	$\alpha - \frac{d+1}{2} \in \mathbb{N}$, $\beta - \alpha \in \mathbb{N}_{\geq 1}$, $\gamma - \frac{d}{2} \in \mathbb{N}_{\geq 1}$	Emery and Alegría (2022)
	Matérn	$\alpha > \frac{d}{2}$, $\beta \to +\infty$, $\gamma = \beta + \frac{1}{2}$, $a = 2\beta b$, $b \in \mathbb{R}_{>0}$	Proposition 7, Matérn (1960)
	Exponential	$\alpha = \frac{d+1}{2}$, $\beta \to +\infty$, $\gamma = \beta + \frac{1}{2}$, $a = 2\beta b$, $b \in \mathbb{R}_{>0}$	Proposition 7, Matérn (1960)
	Gaussian	$\alpha \to +\infty$, $\beta/\alpha \to +\infty$, $\gamma = \beta + \frac{1}{2}$, $a = \beta b/\sqrt{\alpha}$, $b \in \mathbb{R}_{>0}$	Proposition 10, Matérn (1960)
	Incomplete gamma	$\alpha > \frac{d}{2}$, $\beta > \alpha$, $\gamma \to +\infty$, $a = b\sqrt{\gamma}$, $b \in \mathbb{R}_{>0}$	Emery and Alegría (2022)
	Gaussian-polynomial	$\alpha - \frac{d}{2} - 1 \in \mathbb{N}$, $\beta > \alpha$, $\gamma \to +\infty$, $a = b\sqrt{\gamma}$, $b \in \mathbb{R}_{>0}$	Proposition 14
	Complementary error	$\alpha = \frac{d+1}{2}$, $\beta > \alpha$, $\gamma \to +\infty$, $a = b\sqrt{\gamma}$, $b \in \mathbb{R}_{>0}$	Gneiting (1999b)
	Hole effect truncated power	$\alpha - \frac{d}{2} - k \in \mathbb{R}_{>0} \setminus \mathbb{N}$, $\beta - \alpha \in \mathbb{N}_{\geq 1}$, $\gamma - \frac{d}{2} - k \in \mathbb{N}_{\geq 1}$	Proposition 2
	Hole effect truncated polynomial	$\alpha - \frac{d+1}{2} - k \in \mathbb{N}$, $\beta - \alpha \in \mathbb{N}_{\geq 1}$, $\gamma - \frac{d}{2} - k \in \mathbb{N}_{\geq 1}$	Proposition 2
$\mathcal{H}$ class	Hole effect Generalized Wendland	$\alpha > \frac{d}{2} + k$, $\beta - \alpha \geq \frac{\alpha}{2}$, $\gamma = \beta + \frac{1}{2}$	Proposition 4, Emery et al. (2026)
	Hole effect ordinary Wendland	$\alpha - \frac{d+1}{2} - k \in \mathbb{N}$, $\beta - \alpha \geq \frac{\alpha}{2}$, $\gamma = \beta + \frac{1}{2}$	Proposition 4, Emery et al. (2026)
	Hole effect original Wendland	$\alpha - \frac{d+1}{2} - k \in \mathbb{N}$, $2(\beta - \alpha) \in \mathbb{N}_{\geq \alpha}$, $\gamma = \beta + \frac{1}{2}$	Proposition 4, Emery et al. (2026)
($k \in \mathbb{N}_{\geq 1}$)	Hole effect Askey	$\alpha = \frac{d+1}{2} + k$, $\beta - \alpha \geq \frac{\alpha}{2}$, $\gamma = \beta + \frac{1}{2}$	Proposition 6, Emery et al. (2026)
(Theorem 1)	Hole effect Matérn	$\alpha > \frac{d}{2} + k$, $\beta \to +\infty$, $\gamma = \beta + \frac{1}{2}$, $a = 2\beta b$, $b \in \mathbb{R}_{>0}$	Proposition 8, Emery et al. (2026)
	Hole effect Gaussian	$\alpha \to +\infty$, $\beta/\alpha \to +\infty$, $\gamma = \beta + \frac{1}{2}$, $a = \beta b/\sqrt{\alpha}$, $b \in \mathbb{R}_{>0}$	Proposition 12
	Schoenberg	$\alpha = \frac{d+1}{2} + 2k$, $k \to +\infty$, $\frac{\beta}{k} \to +\infty$, $\gamma = \beta + \frac{1}{2}$, $a = 2\beta b$, $b \in \mathbb{R}_{>0}$	Proposition 9, Schoenberg (1938)
	Cosine	$\alpha = 2k + 1$, $k \to +\infty$, $\frac{\beta}{k} \to +\infty$, $\gamma = \beta + \frac{1}{2}$, $a = 2\beta b$, $d = 1$, $b \in \mathbb{R}_{>0}$	Proposition 9, Yaglom (1987)
	Cardinal sine	$\alpha = 2k + 2$, $k \to +\infty$, $\frac{\beta}{k} \to +\infty$, $\gamma = \beta + \frac{1}{2}$, $a = 2\beta b$, $d = 3$, $b \in \mathbb{R}_{>0}$	Proposition 9, Yaglom (1987)
	Hole effect incomplete gamma	$\alpha > \frac{d}{2} + k$, $\beta > \alpha$, $\gamma \to +\infty$, $a = b\sqrt{\gamma}$, $b \in \mathbb{R}_{>0}$	Proposition 14

Table 2: Special cases in Φ_d from the class $\mathcal{H}$.

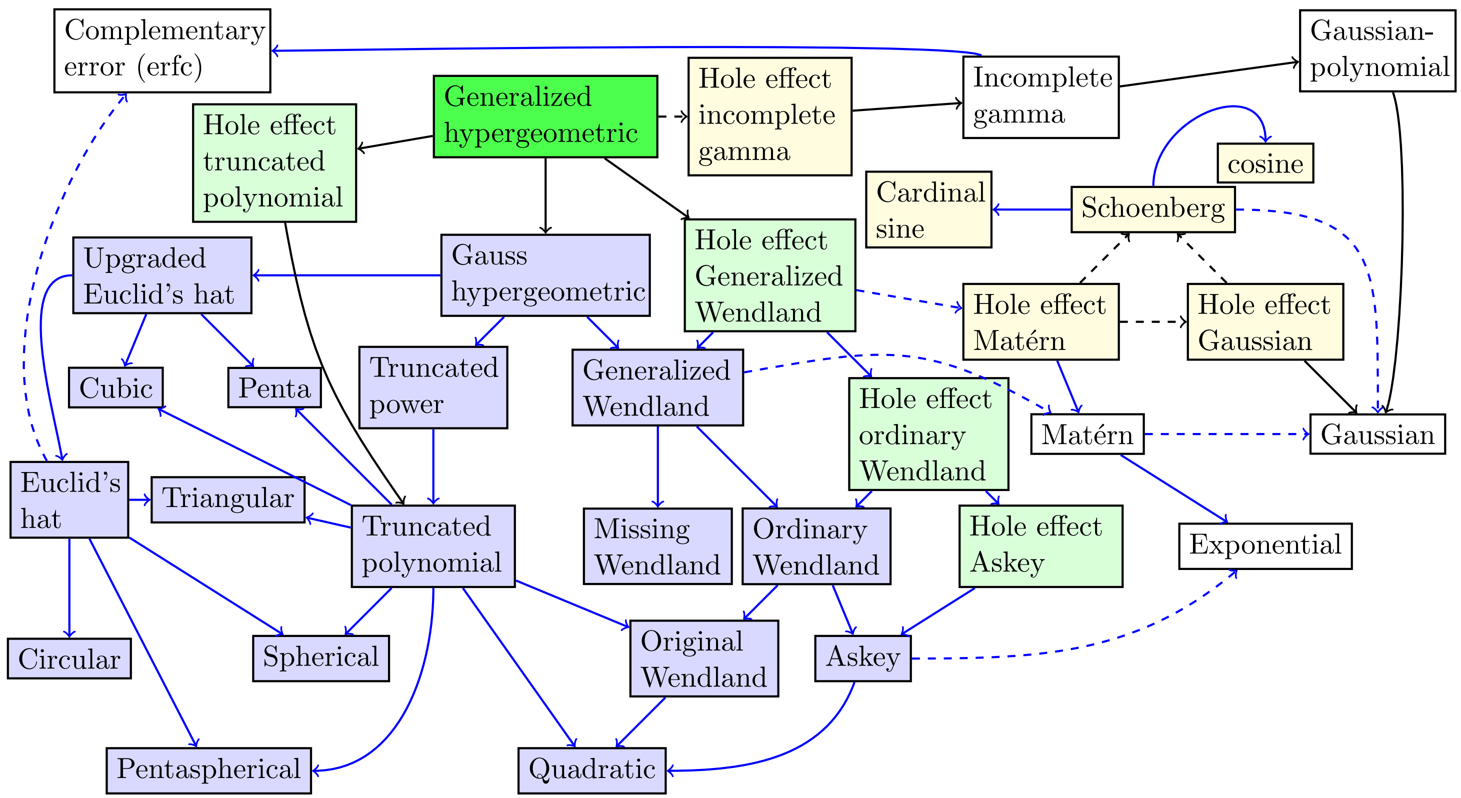

Figure 1: Connections between covariance kernels. Blue boxes are compactly supported kernels; yellow boxes are hole effect kernels; green boxes are compactly supported hole effect kernels. Solid arrows indicate particular cases; dashed arrows indicate asymptotic cases. Connections established in previous literature are indicated in blue; connections proved in this paper are indicated in black.

Some properties of the generalized hypergeometric kernel $\mathcal{H}_{\boldsymbol{\theta}}$ follow.

- *Support.* $\mathcal{H}_{\boldsymbol{\theta}}$ is compactly supported, as it vanishes outside the interval $[0, a)$.
- *Invariance under isotropic scaling.* The class $\mathcal{H}$ is invariant under a scaling of the compact support a, that is, $\mathcal{H}_{\boldsymbol{\theta}}(h) = \mathcal{H}_{\boldsymbol{\theta}_0}(\frac{h}{a})$ for any $h \geq 0$ and $\boldsymbol{\theta}_0 = (1, \alpha, \beta, \gamma, d, k)$.
- *Hole effect.* $\mathcal{H}_{\boldsymbol{\theta}}$ is nonnegative and monotonic when $k = 0$ (Emery and Alegría, 2022), but attains negative values when $k > 0$, as shown in Appendix A and illustrated next.
- *Smoothness.* By using formula 16.3.1 in Olver et al. (2010), one finds the first- and second-order right derivatives of $\mathcal{H}_{\boldsymbol{\theta}}$ at $h = 0$:

$$\left.\frac{\partial_+ \mathcal{H}_{\boldsymbol{\theta}}(h)}{\partial h}\right|_{h=0} = \begin{cases} 0 \text{ if } 2\alpha > d + 2k + 1 \\ -\frac{2\Gamma(\alpha)\Gamma(\beta-\alpha+\frac{1}{2})\Gamma(\gamma-\alpha+\frac{1}{2})\Gamma(\frac{d}{2})}{a\Gamma(\alpha-\frac{1}{2})\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)\Gamma(\frac{d+1}{2})} \text{ if } 2\alpha = d + 2k + 1 \\ -\infty \text{ if } 2\alpha < d + 2k + 1. \end{cases}$$

$$\left.\frac{\partial_+^2 \mathcal{H}_{\boldsymbol{\theta}}(h)}{\partial h^2}\right|_{h=0} = \begin{cases} \frac{4(\frac{d}{2}+k)(1+\frac{d}{2}+k-\beta)(1+\frac{d}{2}+k-\gamma)}{(1+\frac{d}{2}+k-\alpha)da^2} \text{ if } 2\alpha > d + 2k + 2 \\ +\infty \text{ if } 2\alpha < d + 2k + 2. \end{cases}$$

  Note that the case $2\alpha = d + 2k + 2$ is excluded by condition (A.4).

  Accordingly, the parameter $\alpha - k$ controls the regularity of $\mathcal{H}_{\boldsymbol{\theta}}$ at the origin. When $\alpha - k > \frac{d}{2} + 1$, $h \mapsto \mathcal{H}_{\boldsymbol{\theta}}(h)$ is twice differentiable on the right at $h = 0$ and is associated with a Gaussian random field that is mean square differentiable in space.

  More generally, by expanding the generalized hypergeometric function ${}_3F_2$ in (13) into a power series, one obtains an expansion of $\mathcal{H}_{\boldsymbol{\theta}}(h)$ whose most irregular term is $h^{2\alpha-d-2k}$, where the exponent $2\alpha - d - 2k$ is not an even integer due to condition (A.4). The function $\mathcal{H}_{\boldsymbol{\theta}}$ therefore admits finite right derivatives at $h = 0$ up to order $\lfloor 2\alpha - d - 2k \rfloor$, with the odd-order derivatives being zero up to order $\lceil 2\alpha - d - 2k - 1 \rceil$. This implies that $\mathcal{H}_{\boldsymbol{\theta}}$ is associated with a Gaussian random field that is $\lfloor \alpha - \frac{d}{2} - k \rfloor$-times mean square differentiable in space.
- *Behavior near the range.* $\mathcal{H}_{\boldsymbol{\theta}}$ is continuous on $[0, +\infty)$ and infinitely differentiable on $(0, a) \cup (a, +\infty)$. In particular, it is continuous at $h = a$, but may not be differentiable at this particular point. A sufficient condition for $\mathcal{H}_{\boldsymbol{\theta}}$ to be p-times differentiable at $h = a$ is that $\beta + \gamma - \alpha - 2k - \frac{d}{2} - 1 > p$; this condition is also necessary when $\beta + \gamma \notin \mathbb{N}$ and $\beta + \gamma - \alpha - \frac{d}{2} \notin \mathbb{N}$ (Appendix B).
- *Representation as an autoconvolution.* Let $\boldsymbol{\theta}$ fulfilling conditions (A.1) to (A.4) and let K be the stationary kernel in $\mathbb{R}^d$ such that $\mathsf{K}(\boldsymbol{x} - \boldsymbol{x}') = \mathcal{H}_{\boldsymbol{\theta}}(\|\boldsymbol{x} - \boldsymbol{x}'\|_d)$. Then K can be written as a transitive covariogram if, and only if, either $d \leq 2$ or k is even:

$$\mathsf{K}(\boldsymbol{h}) = \int_{\mathbb{R}^d} f(\boldsymbol{u})\overline{f}(\boldsymbol{u} + \boldsymbol{h}) \mathrm{d}\boldsymbol{u}, \quad \boldsymbol{h} \in \mathbb{R}^d,$$

where f is a complex-valued square-integrable function in $\mathbb{R}^d$ that is compactly supported ($f(\boldsymbol{u}) = 0$ for $\|\boldsymbol{u}\| > \frac{a}{2}$) and, if k is even, radially symmetric. This result is a consequence of Theorems 2.1 and 3.1 of Ehm et al. (2004) and the fact that the d-radial spectral density $\widehat{\mathcal{H}}_{\boldsymbol{\theta}}$ can be extended to an entire function on $\mathbb{C}$ that has no purely imaginary zero, except the origin that is a zero of order $2k$:

$$\widehat{\mathcal{H}}_{\boldsymbol{\theta}}(\mathrm{i}u) = (-1)^k \widehat{\varpi}\, a^{d+2k}\, u^{2k}{}_1F_2\left(\begin{matrix}\alpha \\ \beta,\ \gamma\end{matrix}; \frac{a^2u^2}{4}\right) \neq 0 \text{ if } u \in \mathbb{R} \smallsetminus \{0\}.$$

- *Continuation.* Formula (13) is undefined if $\alpha - \frac{d}{2} - k \in \mathbb{N}_{\geq 1}$, as it involves the difference of two infinite terms. However, in such a case, a limit kernel belonging to Φ_d can be defined by continuation (proof in Appendix B):

$$\mathcal{H}_{\boldsymbol{\theta}}(h) = \lim_{\varepsilon \to 0^-} \mathcal{H}_{\boldsymbol{\theta}+(0,\varepsilon,0,0,0,0)}(h), \quad h \geq 0, \quad \alpha - \frac{d}{2} - k \in \mathbb{N}_{\geq 1}. \tag{15}$$

- *Fractal dimension.* Owing to Tauberian theorems, a Gaussian random field with covariance kernel $\mathcal{H}_{\boldsymbol{\theta}}$ has realizations with fractal dimension $D = d + 1 - \frac{\vartheta}{2}$ whenever $0 < \vartheta = 2(\alpha - k) - 1 \leq 2$. To prove it, one just needs to observe that $\widehat{\mathcal{H}}_{\boldsymbol{\theta}}(u)$ behaves like $u^{-\vartheta-1}$ as $u \to +\infty$ (see proof of Theorem 3).

4. Special Cases Through Exact Parameterization

Proposition 1 (Gauss hypergeometric kernel) *Let $\boldsymbol{\theta} = (a, \alpha, \beta, \gamma, d, 0)$ satisfying conditions (A.1) to (A.4) as per Theorem 1. Then, $\mathcal{H}_{\boldsymbol{\theta}}$ is the Gauss hypergeometric kernel introduced by Emery and Alegría (2022):*

$$\mathcal{H}_{\boldsymbol{\theta}}(h) = \frac{\Gamma(\beta - \frac{d}{2})\Gamma(\gamma - \frac{d}{2})}{\Gamma(\beta - \alpha + \gamma - \frac{d}{2})\Gamma(\alpha - \frac{d}{2})} \left(1 - \frac{h^2}{a^2}\right)_+^{\beta-\alpha+\gamma-\frac{d}{2}-1} {}_2F_1\left(\begin{matrix}\beta-\alpha,\ \gamma-\alpha \\ \beta-\alpha+\gamma-\frac{d}{2}\end{matrix}; 1 - \frac{h^2}{a^2}\right). \tag{16}$$

■

The kernel (16) is well defined and belongs to Φ_d even if condition (A.4) does not hold.

Proposition 2 (Hole effect truncated power and hole effect truncated polynomial kernels) *Let $\boldsymbol{\theta} = (a, \alpha, \beta, \gamma, d, k)$ satisfying conditions (A.1) to (A.4) as per Theorem 1, such that $\beta = 1 + \alpha + M$ and $\gamma = 1 + \frac{d}{2} + k + N$ with $M, N \in \mathbb{N}$. Then, one has*

$$\mathcal{H}_{\boldsymbol{\theta}}(h) = \begin{cases} 0 & \textit{if } a \leq h, \\ \sum_{n=0}^{N} \frac{(\frac{d}{2}+k)_n(\frac{d}{2}+k-\alpha-M)_n(-N)_n}{(1+\frac{d}{2}+k-\alpha)_n(\frac{d}{2})_n n!}\left(\frac{h}{a}\right)^{2n} & \\ + \frac{\Gamma(\alpha)\Gamma(1+\alpha+M-\frac{d}{2}-k)\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}+k-\alpha)N!}{\Gamma(\frac{d}{2}+k)\Gamma(\alpha-\frac{d}{2}-k)\Gamma(1+\frac{d}{2}+k+N-\alpha)\Gamma(\alpha-k)M!} & \\ \times \sum_{n=0}^{M} \frac{(\alpha)_n(-M)_n(\alpha-\frac{d}{2}-k-N)_n}{(1+\alpha-\frac{d}{2}-k)_n(\alpha-k)_n n!}\left(\frac{h}{a}\right)^{2n+2\alpha-d-2k} & \textit{if } 0 \leq h < a. \end{cases} \tag{17}$$

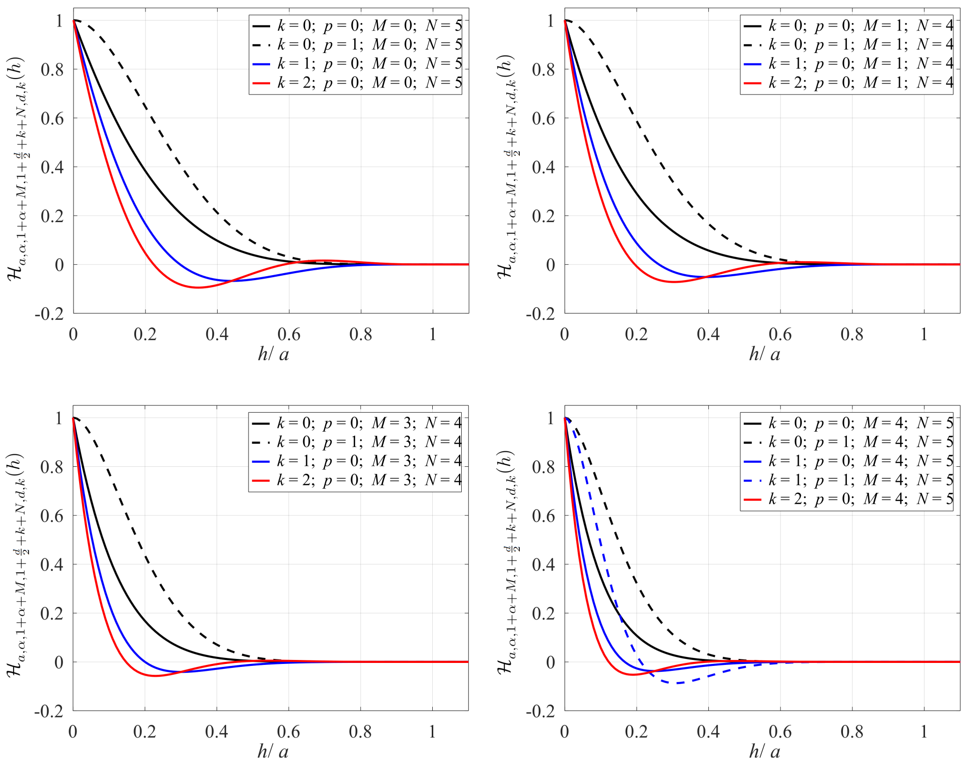

Figure 2: Hole effect truncated polynomial kernel $\mathcal{H}_{\boldsymbol{\theta}}$ for $\boldsymbol{\theta} = (a, \frac{d+1}{2} + k + p, 1 + \frac{d+1}{2} + k + p + M, 1 + \frac{d}{2} + k + N, d, k)$, $d = 2$ and different choices of k, p, M and N.

If, additionally, $2\alpha - d$ is an odd integer, $\mathcal{H}_{\boldsymbol{\theta}}$ reduces to a truncated polynomial function. The related permissibility conditions (A.1) to (A.4) become

- $\alpha = \frac{d+1}{2} + k + p$ *with* $p \in \mathbb{N}$;
- $(1 + M)(2N - d - 2k - 4p) \geq \frac{d+1}{2} + k + p$;
- $M + N - k - 2p \geq \frac{d-1}{2}$. ■

Figure 2 gives examples of such a truncated polynomial kernel (17) with $2\alpha - d$ an odd integer, for $d = 2$, $p \leq 1$ and several choices of the other parameters k, M and N. One observes that the behavior at the origin gets more regular as p increases, which corresponds to a Gaussian random field getting smoother in space. A hole effect emerges when k is positive, while the case $k = 0$ provides monotonic mappings. Note that, as k increases, the amplitude of the hole effect also increases.

Proposition 3 (Ordinary and Generalized Wendland kernels) *The Generalized Wendland kernel (Zastavnyi, 2006) is a special case of the Gauss hypergeometric kernel (16), for which one has* $\mathcal{W}_{a,\xi,\nu,d,0} = \mathcal{H}_{\boldsymbol{\theta}}$ *with* $\boldsymbol{\theta} = (a, \xi + \frac{d+1}{2}, \xi + \frac{d+\nu+1}{2}, \xi + \frac{d+\nu}{2} + 1, 0)$*. In particular,*

$$\mathcal{W}_{a,\xi,\nu,d,0}(h) = \frac{\Gamma(\xi + \frac{\nu+1}{2})\Gamma(\xi + \frac{\nu}{2} + 1)\left(1 - \frac{h^2}{a^2}\right)_+^{\xi+\nu}}{\Gamma(\xi + \nu + 1)\Gamma(\xi + \frac{1}{2})} \, {}_2F_1\left(\begin{matrix}\frac{\nu}{2}, \frac{\nu+1}{2} \\ \xi+\nu+1\end{matrix}; \left(1 - \frac{h^2}{a^2}\right)_+\right), \tag{18}$$

with $\xi > -\frac{1}{2}$ *and* $\nu \geq \nu_{\min}(\xi, d)$*, where* $\nu_{\min}(\xi, d) := \begin{cases} \frac{\sqrt{8\xi+9}-1}{2} & \text{if } d = 1 \text{ and } -\frac{1}{2} < \xi < 0 \\ \xi + \frac{d+1}{2} & \text{otherwise.} \end{cases}$

■

The case when ξ is an integer is known as the *ordinary* Wendland kernel, for which a closed-form expression is available (Hubbert, 2012; Bevilacqua et al., 2024). The subcase when both ξ and ν are integers yields the so-called *original* Wendland kernel, which has a polynomial expression in the interval $[0, a]$ (Wendland, 1995). The case when ξ is a half-integer and ν is an integer is known as the *missing* Wendland kernel (Schaback, 2011), which also has a closed-form expression (Bevilacqua et al., 2024).

Proposition 4 (Hole effect Generalized Wendland kernel) *The hole effect Generalized Wendland kernel* $\mathcal{W}_{a,\xi,\nu,d,k}$ *(Emery et al., 2026) is a particular case of the generalized hypergeometric kernel, for which one has*

$$\mathcal{W}_{a,\xi,\nu,d,k}(h) := \mathcal{H}_{a,\xi+\frac{d+1}{2}+k,\xi+\frac{d+\nu+1}{2}+k,\xi+\frac{d+\nu}{2}+k+1,d,k}(h), \tag{19}$$

with $k \in \mathbb{N}$*,* $\xi > -\frac{1}{2}$ *and* $\nu \geq \nu_{\min}(\xi, d + 2k)$*.* ■

Note that $\mathcal{W}_{a,\xi,\nu,d,k}$ reduces to the Generalized Wendland kernel (18) if $k = 0$. Also, if $\xi + \frac{1}{2} \in \mathbb{N}$, one has to consider the continuation (15) of the generalized hypergeometric kernel in (19). Closed-form expressions of $\mathcal{W}_{a,\xi,\nu,d,k}$ can be obtained when $\xi \in \mathbb{N}$, which yields a *hole effect ordinary Wendland* kernel, see Emery et al. (2026).

Proposition 5 (Askey kernel) *The Askey kernel* $h \mapsto \left(1 - \frac{h}{a}\right)_+^{\nu}$ *(Golubov, 1981) is a particular case of the generalized hypergeometric kernel, corresponding to* $\mathcal{W}_{a,0,\nu,d,0}$ *with* $\nu \geq \frac{d+1}{2}$*.*

■

Proposition 6 (Hole effect Askey kernel) *The hole effect Askey kernel (Emery et al., 2026) is a particular case of the generalized hypergeometric kernel, corresponding to* $\mathcal{W}_{a,0,\nu,d,k}$ *with* $\nu \geq \frac{d+1}{2} + k$*.* ■

5. Special Cases Through Parametric Convergence

The generalized hypergeometric kernel also converges asymptotically to globally supported kernels, as indicated next.

5.1 Matérn-like Kernels

The following result is of independent interest and provides a parameterization of the Generalized Wendland kernel that includes the Matérn kernel as a limit case.

Proposition 7 (Matérn kernel) *Let $a, \mu, \nu \in \mathbb{R}_{>0}$ and $d \in \mathbb{N}_{\geq 1}$. As μ tends to $+\infty$, the Generalized Wendland kernel $\mathcal{W}_{\mu a, \nu - \frac{1}{2}, \mu, d, 0}$ converges uniformly on $[0, +\infty)$ to the Matérn kernel $\mathcal{M}_{a,\nu,d}$.* ■

Proposition 8 (Hole effect Matérn kernel) *Let $a, \mu, \nu \in \mathbb{R}_{>0}$, $d \in \mathbb{N}_{\geq 1}$ and $k \in \mathbb{N}$. As μ tends to $+\infty$, the hole effect Generalized Wendland kernel $\mathcal{W}_{\mu a, \nu - \frac{1}{2}, \mu, d, k}$ converges uniformly on $[0, +\infty)$ to the hole effect Matérn kernel defined through (Emery et al., 2026)*

$$\mathcal{M}_{a,\nu,d,k}(h) := \begin{cases} \sum_{q=0}^{k} \sum_{r=0}^{\max\{0,q-1\}} \sum_{s=0}^{q-r} \sum_{t=0}^{q-r-s} \left(\frac{h}{a}\right)^{\nu+q-r-s} K_{\nu+2t+r+s-q}\left(\frac{h}{a}\right) \\ \quad \times \frac{(-1)^{q-s}(q-r)!(q-r)_r(\nu+1-s)_s(k-q+1)_q(q)_r}{2^{\nu+2q-s-1} q!\, r!\, s!\, t!\, (q-r-s-t)!\, \Gamma(\nu) (\frac{d}{2})_q} \quad \text{if } h > 0 \\ 1 \quad \text{if } h = 0. \end{cases} \tag{20}$$

■

As an illustration, the convergence of $\mathcal{W}_{\mu a, \nu - \frac{1}{2}, \mu, d, k}$ to $\mathcal{M}_{a,\nu,d,k}$ as μ tends to $+\infty$ can be appreciated in Figure 3 when $k = 0$ or 2, for $d = 2$ and specific values of the parameters.

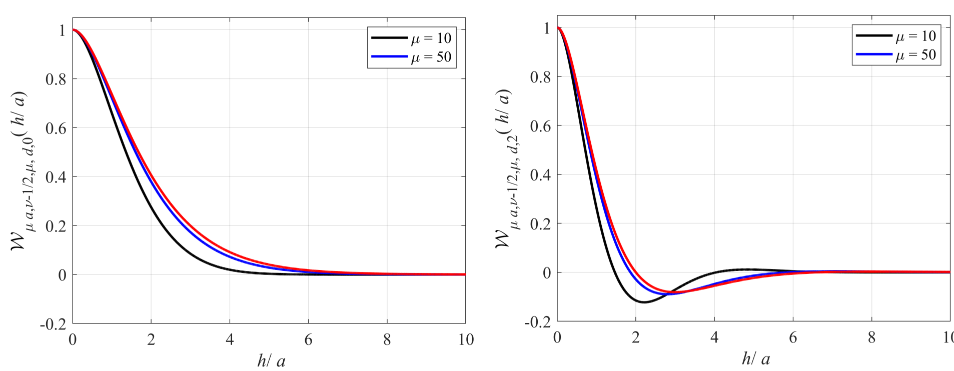

Figure 3: $\mathcal{W}_{\mu a, \nu - \frac{1}{2}, \mu, d, k}$ and $\mathcal{M}_{a,\nu,d,k}$ (red line) when $\mu = 10, 50$, $\nu = 1.5$, $d = 2$ and $k = 0$ (left) or $k = 2$ (right).

If $k = 0$, then one recovers the traditional Matérn kernel (11). Another special case is obtained when ν is a half-integer, in which case the modified Bessel functions can be expressed in terms of exponential and power functions (Gradshteyn and Ryzhik, 2007, 8.468). Analytical expressions of $\mathcal{M}_{a,\nu,d,k}$ in terms of special functions can be found in Emery et al. (2026).

For $\nu \in \mathbb{N}_{\geq 1}$, Proposition 8 still holds by considering the continuation of the hole effect Generalized Wendland kernel (Emery et al., 2026), while the hole effect Matérn kernel remains given by (20).

5.2 Schoenberg Kernel

When k and ν increase at the same time, the kernels $\mathcal{W}_{\mu a,\nu-\frac{1}{2},\mu,d,k}$ and $\mathcal{M}_{a,\nu,d,k}$ behave as differentiable (on the right at the origin) oscillating correlation functions. In particular, the following result establishes the convergence of these kernels to the Schoenberg kernel (6) as k tends to infinity, which is an infinitely differentiable and oscillating correlation function that has infinitely many zeros (Chilès and Delfiner, 2012).

Proposition 9 (Schoenberg kernel) *Let $a, \mu \in \mathbb{R}_{>0}$, $d \in \mathbb{N}_{\geq 1}$ and $k \in \mathbb{N}$. As both k and $\frac{\mu}{k}$ tend to $+\infty$, the hole effect Generalized Wendland kernel $\mathcal{W}_{\mu a,k,\mu,d,k}$ and the hole effect Matérn kernel $\mathcal{M}_{a,k+\frac{1}{2},d,k}$ converge uniformly on any bounded interval of $[0, +\infty)$ to the Schoenberg kernel $\mathcal{J}_{a,d}$.* ■

Particular cases of Schoenberg kernels include the cosine and cardinal sine kernels, for $d = 1$ and $d = 3$, respectively (Chilès and Delfiner, 2012).

As an illustration, Figure 4 depicts the $\mathcal{M}_{a,k+\frac{1}{2},d,k}$ kernel for $d = 2$ and $k = 5, 10, 100$ and the Schoenberg kernel $\mathcal{J}_{a,d}$. The former kernels tend to the latter as k increases.

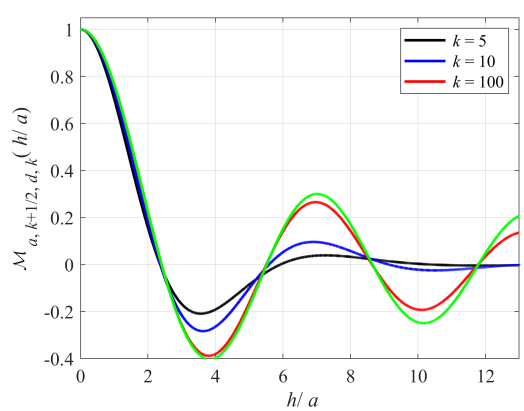

Figure 4: $\mathcal{M}_{a,k+\frac{1}{2},d,k}$ when $d = 2$ and $k = 5, 10, 100$ and $\mathcal{J}_{a,d}$ (green line).

5.3 Gaussian-like Kernels

We also have convergence results to the well-known Gaussian kernel, which belongs to Φ_∞ and is defined as

$$\mathcal{G}_a(h) = \exp\left(-\frac{h^2}{a^2}\right), \quad h \geq 0, \quad a > 0. \tag{21}$$

Proposition 10 (Gaussian kernel) *Let $a, \mu, \nu \in \mathbb{R}_{>0}$ and $d \in \mathbb{N}_{\geq 1}$. As both ν and $\mu\nu^{-2}$ tend to $+\infty$, the Generalized Wendland kernel $\mathcal{W}_{\mu a/\sqrt{4\nu},\nu-\frac{1}{2},\mu,d,0}$ uniformly converges on $[0,+\infty)$ to the Gaussian kernel $\mathcal{G}_a$.* ■

Note that both the Matérn and Schoenberg kernels also uniformly converge to the Gaussian kernel under a suitable parameterization, see Stein (1999) and Schoenberg (1938).

Proposition 11 (Hole effect Gaussian kernel) *Let $a \in \mathbb{R}_{>0}$, $d \in \mathbb{N}_{\geq 1}$ and $k \in \mathbb{N}$. Define the hole effect Gaussian kernel as*

$$\mathcal{G}_{a,d,k}(h) := \frac{\Gamma(\frac{d}{2})k!}{\Gamma(\frac{d}{2}+k)} \exp\left(-\frac{h^2}{4a^2}\right) L_k^{\frac{d}{2}-1}\left(\frac{h^2}{4a^2}\right), \quad h \geq 0, \tag{22}$$

with the generalized Laguerre polynomial $L_k^{\frac{d}{2}-1}$ given by (Olver et al., 2010, 5.5.3 and 18.5.12)

$$L_k^{\frac{d}{2}-1}(x) = \sum_{n=0}^{k} \frac{(\frac{d}{2}+n)_{k-n}(-x)^n}{(k-n)!n!} = \frac{\Gamma(k+\frac{d}{2})}{k!} \sum_{n=0}^{k} \frac{(-k)_n x^n}{\Gamma(n+\frac{d}{2})n!}, \quad x \in \mathbb{R}. \tag{23}$$

Then $\mathcal{G}_{a,d,k}$ belongs to Φ_d. ■

The generalized Laguerre polynomial (23) has k different zeros that are positive (Szegö, 1975), then so does $\mathcal{G}_{a,d,k}$: since it asymptotically tends to 0 at infinity, this kernel has a hole effect with k waves. Also note that (21) is a particular case of (22), as $\mathcal{G}_{a,d,0} = \mathcal{G}_a$.

Proposition 12 *Let $a, \mu \in \mathbb{R}_{>0}$, $d, n \in \mathbb{N}_{\geq 1}$ and $k \in \mathbb{N}$. As both n and $\frac{\mu}{n}$ tend to $+\infty$, $\mathcal{W}_{\mu a/\sqrt{4n},n,\mu,d,k}$ and $\mathcal{M}_{a/\sqrt{4n},n+\frac{1}{2},d,k}$ uniformly converge on any bounded interval of $[0,+\infty)$ to $\mathcal{G}_{a,d,k}$.* ■

Proposition 13 *Let $a \in \mathbb{R}_{>0}$, $d \in \mathbb{N}_{\geq 1}$ and $k \in \mathbb{N}$. As k tends to $+\infty$, $\mathcal{G}_{a\sqrt{k},d,k}$ uniformly converges on any bounded interval of $[0,+\infty)$ to $\mathcal{J}_{a,d}$.* ■

5.4 Incomplete Gamma Kernels

Proposition 14 (Hole effect incomplete gamma kernel) *Let* $\boldsymbol{\theta} = (a\sqrt{\gamma}, \alpha, \beta, \gamma, d, k)$ *satisfying conditions (A.1) to (A.4) of Theorem 1. As* γ *tends to* $+\infty$*,* $\mathcal{H}_{\boldsymbol{\theta}}$ *uniformly converges on any bounded interval of* $[0, +\infty)$ *to the hole effect incomplete gamma kernel* $\mathcal{I}_{a,\alpha,d,k}$*, defined by*

$$\mathcal{I}_{a,\alpha,d,k}(h) = 1 - \sum_{n=0}^{k} \frac{(-1)^n k!(\alpha - k + n)_{k-n}}{n!(k-n)!\Gamma(\alpha - \frac{d}{2} - k)\,(\frac{d}{2})_k} \Gamma^{-}\left(\alpha - \frac{d}{2} - k + n, \frac{h^2}{a^2}\right), \quad h \geq 0.$$

■

The name of the kernel is due to the fact that $\mathcal{I}_{a,\alpha,d,0}$ is the regularized incomplete gamma kernel introduced by Emery and Alegría (2022):

$$\mathcal{I}_{a,\alpha,d,0}(h) = 1 - \frac{1}{\Gamma(\alpha - \frac{d}{2})} \Gamma^{-}\left(\alpha - \frac{d}{2}, \frac{h^2}{a^2}\right) = \frac{1}{\Gamma(\alpha - \frac{d}{2})} \Gamma^{+}\left(\alpha - \frac{d}{2}, \frac{h^2}{a^2}\right), \quad h \geq 0.$$

Particular cases, which all belong to Φ_∞ insofar as the expression of these kernels does not depend on d, include

1. The Gaussian kernel when $\alpha = \frac{d}{2} + 1$ (Olver et al., 2010, 8.4.8):
$$\mathcal{I}_{a,\frac{d}{2}+1,d,0}(h) = \mathcal{G}_a(h), \quad h \geq 0.$$

2. A Gaussian-polynomial kernel when $\alpha = \frac{d}{2} + 1 + q$ with $q \in \mathbb{N}$ (Olver et al., 2010, 8.4.8):
$$\mathcal{I}_{a,\frac{d}{2}+1+q,d,0}(h) = \exp\left(-\frac{h^2}{a^2}\right) \sum_{n=0}^{q} \frac{1}{n!} \left(\frac{h^2}{a^2}\right)^n, \quad h \geq 0.$$

3. The complementary error function when $\alpha = \frac{d+1}{2}$ (Olver et al., 2010, 8.4.6):
$$\mathcal{I}_{a,\frac{d+1}{2},d,0}(h) = \operatorname{erfc}\left(\frac{h}{a}\right), \quad h \geq 0.$$

   Gneiting (1999b) proved that $\mathcal{H}_{\boldsymbol{\theta}}$ with $\boldsymbol{\theta} = (a(\frac{d-1}{2})^{1/2}, \frac{d+1}{2}, \frac{d}{2}+1, d+1, d, 0)$ uniformly converges on $[0, +\infty)$ to $\mathcal{I}_{a,\frac{d+1}{2},d,0}$ as d tends to infinity.

6. Sobolev Consequences Under the Class $\mathcal{H}$

Theorem 3 has implications in many branches of statistics, machine learning, and approximation theory. We describe some of them.

1. Best linear unbiased prediction under a misspecified covariance kernel is an important subject in spatial statistics (Stein, 2002, 2011) and approximation theory (Scheuerer, 2010). Typically, the performance of the kriging predictor under an incorrect class of covariance kernels is measured by comparison with the *true* kernel under fixed domain asymptotics, that is, considering observations that increase over a compact set in such a way that the distance between the observations tends to zero. Stein (2002) proved that the equivalence of Gaussian measures is a sufficient condition to ensure asymptotic optimality of the kriging predictor under a misspecified covariance kernel. The Sobolev properties of the kernel are of crucial importance and have been used under this framework for the Matérn class (Zhang, 2004) as well as for the Generalized Wendland class (Bevilacqua et al., 2019). This work fixes the basis to understand optimal unbiased linear prediction for a wealth of kernels that have not been studied so far under this perspective.

2. The *screening effect* is also a well known problem in spatial statistics. It is used to describe a situation where the interpolant depends mostly on those observations that are located nearest to the predictand (Stein, 2002). Such a problem has been of interest to geostatisticians for decades (Chilès and Delfiner, 2012) because it translates into the optimality property of reducing considerably the computational burden associated with the kriging predictor when handling large data sets. Quantifying screening effects under a specified class of kernels is a major task that relies on several aspects, such as the spatial design (how to locate the observation points), the dimension of the Euclidean space where the spatial domain is embedded, the covariance kernel attached to a Gaussian random field (or, equivalently, its spectral density) and the mean-square differentiability in all directions of the random field. We first note that the screening effect is often quantified in a very practical way (Chilès and Delfiner, 2012). A formalization of the same is due to Stein (1999), who provided sufficient conditions for the screening effect to happen under a regular sampling design. Stein (2011) conjectured that, under the spectral condition
$$\lim_{\|\boldsymbol{\omega}\|\to\infty} \sup_{\|\boldsymbol{\tau}\|<R} \left| \frac{\hat{\mathsf{K}}(\boldsymbol{\omega}+\boldsymbol{\tau})}{\hat{\mathsf{K}}(\boldsymbol{\omega})} - 1 \right| = 0, \tag{24}$$
one has screening effect under an irregular asymptotic design, and showed that (24) is verified for $d \le 2$ with mean-square continuous but nondifferentiable Gaussian random fields, under some specific designs. Porcu et al. (2020) proved that (24) holds for the Generalized Wendland kernel, the argument being based on the tails of the related spectrum, and hence on the Sobolev properties. By following this argument, it becomes straightforward to deduce that such a condition is verified for the class $\mathcal{H}_{\boldsymbol{\theta}}$ as well.

3. Theoretical results related to Gaussian regression in machine learning are strongly connected to the Sobolev properties, and we refer the reader to Korte-Stapff et al.

(2025). For example, Sobolev smoothness is of crucial importance in Bayesian contraction rates (van der Vaart and van Zanten, 2011). Similar results where Sobolev rates pop up are contained in Schaback and Wendland (2006), Scheuerer et al. (2013) and Narcowich et al. (2006). As mentioned in Korte-Stapff et al. (2025) (see the references therein), the Sobolev properties turn to be fundamental within uncertainty quantification in nonparametric methods. The popular maximum mean discrepancies (Oates et al., 2017) have been coupled with kernel methods, for which Sobolev properties cover a fundamental role, and the reader is referred to the most recent contribution in this direction by Barp et al. (2022). The class of kernels proposed in this paper is a very good candidate in all these directions. Additionally, the property of compact support allows for considerable computational gains while preserving the required smoothness properties. Hence, extension of the previously mentioned directions to this class becomes imperative. It is not clear to the authors how the *hole effect* will play (if any) role within these research direction, although this aspect deserves attention.

4. Kernel methods have been widely used in the last decade to solve some systems of partial differential equations (PDEs) that were originally proposed within the approximation theory framework by Fasshauer (1997), and for which a Bayesian turnaround was provided by Cockayne et al. (2019). The very interesting connection between statistics and approximation theory is provided by two facts: (a) the conditional mean of the process constructed by Cockayne et al. (2019) coincides with the *symmetric collocation* method introduced by Fasshauer (1997), and (b) the conditional variance is actually a measure of uncertainty quantification for the solution, while allowing for a finite computational budget. Sobolev methods become important because implementation of these methods requires regularity of a given order for the paths of the associated Gaussian field. Oversmoothness would affect accuracy in uncertainty quantification. The ability to customize smoothness has normally been attributed to the Matérn class, which satisfies a specific class of stochastic partial differential equations (SPDEs) and has become especially popular within the SPDE approximation thanks to the masterpiece of Lindgren et al. (2011) and subsequently Bolin and Lindgren (2011) and Bolin and Kirchner (2020). The class $\mathcal{H}$ opens for a wide spectrum analysis of several classes of (S)PDEs in concert with their application to Bayesian computation as a probabilistic extension to meshless methods.

5. The Bayesian community has been increasingly concerned with a specific class of PDE called *Stein equation*, which is used to compute the posterior expectation of a given function. Numerical solutions of this equation involve kernel methods (Oates et al., 2017; South et al., 2022). Again, there is no surprise that smoothness plays a major role. Using the $\mathcal{H}$ class as a surrogate to the Matérn kernel would allow, within this context, for computationally cheaper solutions while preserving the customizable smoothness property.

7. Relevance and Impact on the Machine Learning Community

The class $\mathcal{H}$ of generalized hypergeometric kernels introduced in this paper provides a compelling contribution to kernel-based machine learning. We have achieved a unifying way to have smoothness control, compact or global support, and a switch from positive to negative dependencies through a unique model. We drive the reader's attention to the impact of this contribution for the machine learning community.

1. *Generalizing Framework for Kernel Learning.* Kernel methods are the foundation of statistical learning and of a wealth of algorithms such as support vector machines, kernel PCA, and Gaussian process (GP) models (Rasmussen and Williams, 2006; Kanagawa et al., 2018). However, the *design* of kernels is largely based on *ad hoc* procedures, with a strong bias towards the Matérn family. The proposed class $\mathcal{H}$ subsumes most classical kernels as special or asymptotic cases, while enriching the portfolio of modeling alternatives. This enables practitioners to interpolate between families, offering both mathematical rigor and application-specific flexibility.

2. *Sobolev Characterization and Learning Theory.* The link between RKHS and Sobolev spaces allows for generalization guarantees, contraction rates in Bayesian regression, and optimal recovery (Schaback and Wendland, 2006; van der Vaart and van Zanten, 2011). The characterization of the native spaces associated with the class $\mathcal{H}$, in terms of Sobolev smoothness, provides a bridge between kernel design and learning theory foundations. This especially applies to function estimation over structured domains such as manifolds or graphs, where smoothness priors influence statistical efficiency and numerical stability (Korte-Stapff et al., 2025).

3. *Scalability and Sparsity through Compact Support.* Computational efficiency is a fundamental problem in large-scale machine learning. It is well known that compactly supported kernels allow for sparse Gram matrices, which in turn eases scalable kernel interpolation, GP regression, and PDE solvers (Wilson and Nickisch, 2015; Bevilacqua et al., 2019). While *ad hoc* truncation methods have been used as a shortcut to guarantee such sparsity, we warn the reader that such methods do not guarantee positive definiteness of the associated Gram matrix. Our approach does not suffer from such a problem. This aspect is central to sparse variational GPs, inducing point methods, and kernel interpolation techniques (Williams and Seeger, 2001; Rahimi and Recht, 2007).

4. *Probabilistic Numerics and Kernel Quadrature.* Probabilistic numerical methods are largely based on kernels to solve deterministic problems (for instance, integration, differential equations) in a Bayesian framework (Cockayne et al., 2019). These solvers have a performance that is largely characterized by the properties—smoothness, support, spectral decay—of their associated kernel. The class proposed in this paper allows for tailored priors for such applications, in turn offering better bias-variance con-

trol in Bayesian quadrature, Stein kernel methods, and meshless collocation (Oates et al., 2017; Barp et al., 2022).

5. *Opening to the Future.* The class $\mathcal{H}$ provides groundwork for next-generation kernel learning, kernel meta-learning, and GP-based Bayesian optimization. We believe that our work provides foundation to the development of kernel architectures that must be tailored to data properties—a rising trend in modern ML. In particular, we believe that our contribution is useful even to spectral methods and numerical PDE solvers, and offers a bridge between learning theory, computational science, and probabilistic modeling.

8. Concluding Remarks

We introduced a versatile family of isotropic kernels—the $\mathcal{H}$ class—that are positive definite in Euclidean spaces and are parameterized by the space dimension d and five scalar parameters $(a, \alpha, \beta, \gamma, k)$. While a controls the support (correlation range), α controls the smoothness (behavior near the origin), β and γ control the shape, in particular, the curvature and the regularity near the range, and k controls the hole effect (number of waves before reaching the range). The smoothness relates to the local properties—Sobolev spaces, fractal dimension and mean square differentiability—of associated Gaussian random fields.

Our kernel attains a wealth of well-known kernels as special or asymptotic cases and, as a by-product, allows to adjust local properties of previously proposed kernels that have not been studied beforehand.

The price to pay for such a generality is that, in statistical applications, some of the parameters may be chosen rather than estimated in order to have competitive performance of the maximum likelihood estimator, both in terms of statistical accuracy and computational cost. Parameter estimation is a broad topic that clearly deserves further investigation.

This work has impact in all the areas of machine learning, statistics, numerical analysis, and approximation theory. We expect these communities to be largely engaged to explore further properties that are notoriously of crucial importance in specific disciplines. In machine learning, this kernel might be taken as a benchmark to study its properties in terms of maximum mean discrepancies under Stein kernel methods. Another important subject within both machine learning and statistics is to explore the properties of this kernel in terms of posterior contraction rates in Gaussian regression (Rosa et al., 2024). In spatial statistics, understanding the properties of the $\mathcal{H}$ class in terms of equivalence of Gaussian measures will have crucial importance as it will allow to understand the interpolation properties (kriging) under a wealth of kernels, generalizing considerably the works of Zhang (2004) and Bevilacqua et al. (2019).

Data Science is providing many challenges, including *fancy* data domains. It will be challenging to have similar constructions to $\mathcal{H}$ for non-Euclidean domains and under different metrics.

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Appendix A. A Turning Bands Construction of the Generalized Hypergeometric Kernel

The turning bands operator (Matheron, 1973) transforms an isotropic kernel defined in $\mathbb{R}^{d+p}$ to another isotropic kernel defined in $\mathbb{R}^d$, with $p \in \mathbb{N}$. Specifically, let $C_{d+p} \in \Phi_{d+p}$ such that $C_{d+p}(\|\cdot\|_{d+p})$ is absolutely integrable in $\mathbb{R}^{d+p}$, and let g_{d+p} be the associated $(d+p)$-radial Schoenberg density. The turning bands operator of order p of C_{d+p} is the mapping $C_d \in \Phi_d$ with d-radial Schoenberg density g_{d+p}. Accounting for (9), this implies (Emery et al., 2026, Lemma 2)

$$C_{d+p}(h) = \frac{2\Gamma(\frac{d+p}{2})}{\Gamma(\frac{p}{2})\Gamma(\frac{d}{2})} h^{2-d-p} \int_0^h u^{d-1}(h^2-u^2)^{\frac{p}{2}-1} C_d(u)\mathrm{d}u, \quad h>0, \tag{25}$$

and

$$\widehat{C}_d(u) = \frac{\pi^{\frac{p}{2}}\Gamma(\frac{d}{2})}{\Gamma(\frac{d+p}{2})} u^p \widehat{C}_{d+p}(u), \quad u>0, \tag{26}$$

where $\widehat{C}_{d+p}$ and $\widehat{C}_d$ are the $(d+p)$- and d-radial spectral densities of C_{d+p} and C_d, respectively.

When $p=2$, one has

$$C_d(h) = \frac{h^{1-d}}{d} \frac{\partial[h^d C_{d+2}(h)]}{\partial h}, \quad h>0. \tag{27}$$

In the general case, when $p=2k$ with $k \in \mathbb{N}$, one can rewrite (25) as

$$x^{\frac{d}{2}+k-1} C_{d+2k}(\sqrt{x}) = \frac{2\Gamma(\frac{d}{2}+k)}{\Gamma(k)\Gamma(\frac{d}{2})} \int_0^{\sqrt{x}} u^{d-1}(x-u^2)^{k-1} C_d(u)\mathrm{d}u$$

and differentiate k times using Gradshteyn and Ryzhik (2007, 0.410, 0.42 and 0.433.1) to obtain (Emery et al., 2026, Lemma 2)

$$C_d(h) = \sum_{q=0}^{k} \sum_{r=0}^{\max\{0,q-1\}} \frac{(-1)^r (k-q+1)_q (q)_r (q-r)_r \, h^{q-r} \, C_{d+2k}^{(q-r)}(h)}{2^{q+r} q! \, r! (\frac{d}{2})_q}. \tag{28}$$

Also, C_{d+2k} and C_d have the same value at the origin: this stems from (5) and the fact that both kernels have, by definition, the same Schoenberg measure. Furthermore, if C_{d+2k} is continuous, nonnegative and supported in $[0,a]$, the following properties are a consequence of (27) and (28):

1. C_d vanish for $h > a$ because so does C_{d+2k}: both kernels are compactly supported.
2. If $C_{d+2k}(0) - C_{d+2k}$ is, up to a positive constant, equivalent to $h \mapsto h^\eta$ (with $\eta \in \mathbb{R}_{>0}$) as $h \to 0^+$, then so does $C_d(0) - C_d$: the two kernels have the same smoothness.
3. C_d has at least k different zeros in $(0,a)$, which results from (27) and a recursive application of Rolle's theorem: hole effects appear when $k > 0$.

In particular, let $\boldsymbol{\theta} = (a, \alpha, \beta, \gamma, d, k)$ satisfying conditions (A.1) to (A.4). Owing to (14) and (26), it is seen that $\mathcal{H}_{\boldsymbol{\theta}}$ is obtained, up to a positive factor, by applying the turning bands operator of order $2k$ to $\mathcal{H}_{\boldsymbol{\theta}'}$ with $\boldsymbol{\theta}' = (a, \alpha, \beta, \gamma, d+2k, 0)$ that also satisfies conditions (A.1) to (A.4). This construction generalizes that of Gneiting (2002), who applied the turning bands operator to a subfamily of ordinary Wendland kernels (the case $\alpha = \frac{d+3}{2}$, $\beta \geq \frac{3\alpha}{2}$, $\gamma = \beta + \frac{1}{2}$ and $k = 1$, which leads to a subcase of the hole effect Generalized Wendland kernel presented in Proposition 4).

Since $\mathcal{H}_{\boldsymbol{\theta}'}$ is continuous, nonnegative and supported in $[0,a]$ (Emery and Alegría, 2022), the aforementioned properties hold. In particular, $\mathcal{H}_{\boldsymbol{\theta}}$ has the same smoothness as $\mathcal{H}_{\boldsymbol{\theta}'}$ and exhibits one or more hole effects as soon as $k > 0$.

Appendix B. Analytical Expressions of the Generalized Hypergeometric Kernel

Expressions in terms of Gauss hypergeometric functions. Using formulae 7.4.1.2 of Prudnikov et al. (1990) and 5.5.3 of Olver et al. (2010), one can rewrite (13) in terms of Gauss hypergeometric functions:

$$\begin{aligned}
\mathcal{H}_{\boldsymbol{\theta}}(h) = &\sum_{n=0}^{k} \frac{(-1)^n k! (1+\frac{d}{2}+k-\beta)_n (1+\frac{d}{2}+k-\gamma)_n}{n!(k-n)!(1-\frac{d}{2}-n)_n (1+\frac{d}{2}+k-\alpha)_n} \\
&\times \left(\frac{h}{a}\right)^{2n} {}_2F_1\left(\begin{matrix} 1+\frac{d}{2}+k-\beta+n,\ 1+\frac{d}{2}+k-\gamma+n \\ 1+\frac{d}{2}+k-\alpha+n \end{matrix} ; \frac{h^2}{a^2} \right) \\
&+ \frac{\Gamma(\beta-\frac{d}{2}-k)\Gamma(\gamma-\frac{d}{2}-k)\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}+k-\alpha)}{\Gamma(\frac{d}{2}+k)\Gamma(\alpha-\frac{d}{2}-k)\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)} \\
&\times \sum_{n=0}^{k} \frac{(-1)^{n+k} k! (1-\alpha)_{k-n} (1+\alpha-\beta)_n (1+\alpha-\gamma)_n}{n!(k-n)!(1+\alpha-\frac{d}{2}-k)_n} \\
&\times \left(\frac{h}{a}\right)^{2\alpha-d-2k+2n} {}_2F_1\left(\begin{matrix} 1+\alpha-\beta+n,\ 1+\alpha-\gamma+n \\ 1+\alpha-\frac{d}{2}-k+n \end{matrix} ; \frac{h^2}{a^2} \right), \quad 0 \leq h < a.
\end{aligned} \tag{29}$$

An alternative is to apply (28) to the Gauss hypergeometric kernel $\mathcal{H}_{\boldsymbol{\theta}'}$ defined in Appendix A. Using (16) with $d+2k$ instead of d, as well as formulae 0.432 of Gradshteyn and Ryzhik (2007) and 15.5.4 of Olver et al. (2010), one finds

$$\begin{aligned}\mathcal{H}_{\boldsymbol{\theta}}(h) = & \sum_{q=0}^{k}\sum_{r=0}^{q}\sum_{s=0}^{\max\{0,q-r-1\}} \frac{(-1)^q (k-q+1)_q (q)_r (q-r)_r (q-r-2s+1)_{2s}}{2^{2r+2s} q!\, r!\, s!\, (\frac{d}{2})_q} \\ & \times \frac{\Gamma(\beta-\frac{d}{2}-k)\Gamma(\gamma-\frac{d}{2}-k)}{\Gamma(\beta-\alpha+\gamma-\frac{d}{2}-k-q+r+s)\Gamma(\alpha-\frac{d}{2}-k)} \\ & \times \left(\frac{h}{a}\right)^{q-r} \left(1-\frac{h}{a}\right)_+^{\beta-\alpha+\gamma-\frac{d}{2}-k-1-s} \left(1+\frac{h}{a}\right)^{\beta-\alpha+\gamma-\frac{d}{2}-k-q+r+s-1} \\ & \times {}_2F_1\left({\beta-\alpha,\ \gamma-\alpha \atop \beta-\alpha+\gamma-\frac{d}{2}-k-q+r+s}; 1-\frac{h^2}{a^2}\right), \quad 0<h\leq a.\end{aligned} \tag{30}$$

The right-hand side of (30) is a continuous function on $(0,a]$ that vanishes at $h=a$ under conditions (A.1) to (A.3), even if condition (A.4) does not hold, which proves that $\mathcal{H}_{\boldsymbol{\theta}}$ can be defined by continuation when $\alpha-\frac{d}{2}$ is an integer. This continuation is still a member of Φ_d, insofar as it is the image by the turning bands operator of order $2k$ of a function ($\mathcal{H}_{\boldsymbol{\theta}'}$) belonging to Φ_{d+2k}.

As the Gauss hypergeometric function ${}_2F_1$ is implemented in the GNU scientific library and in prominent programming languages such as R, Python or Matlab, the expressions (29) and (30) allow a numerically stable computation of $\mathcal{H}_{\boldsymbol{\theta}}$.

Expression in terms of a Meijer function. Using formulae 8.2.2.3 and 8.2.2.15 of Prudnikov et al. (1990), one can rewrite (13) in terms of a Meijer G-function:

$$\mathcal{H}_{\boldsymbol{\theta}}(h) = \begin{cases} 0 \text{ if } h\geq a \\ \frac{\Gamma(\frac{d}{2})\Gamma(\beta-\frac{d}{2}-k)\Gamma(\gamma-\frac{d}{2}-k)}{\Gamma(\alpha-\frac{d}{2}-k)\Gamma(\frac{d}{2}+k)} G_{3,\,3}^{2,\,1}\left(\frac{h^2}{a^2}\middle| {1-\frac{d}{2}-k,\ \beta-\frac{d}{2}-k,\ \gamma-\frac{d}{2}-k \atop 0,\ \alpha-\frac{d}{2}-k,\ 1-\frac{d}{2}}\right) \text{ if } 0<h<a \\ 1 \text{ if } h=0. \end{cases} \tag{31}$$

One can also study the behavior of $\mathcal{H}_{\boldsymbol{\theta}}$ near the range by using the expansion of the Meijer G-function of argument close to 1 (Prudnikov et al., 1990, 8.2.2.60). It comes

$$\mathcal{H}_{\boldsymbol{\theta}}(h) \underset{h\to a^-}{\sim} \varsigma \left(1-\frac{h}{a}\right)^{\beta+\gamma-\alpha-2k-\frac{d}{2}-1} \tag{32}$$

with $\varsigma\neq 0$, provided conditions (A.1) to (A.4) hold, $\beta+\gamma\notin\mathbb{N}$ and $\beta+\gamma-\alpha-\frac{d}{2}\notin\mathbb{N}$. In this setting, $\mathcal{H}_{\boldsymbol{\theta}}$ has left derivatives of orders 1 to p that vanish at $h=a$ (hence, $\mathcal{H}_{\boldsymbol{\theta}}$ is p-times differentiable at $h=a$) if, and only if, $\beta+\gamma-\alpha-2k-\frac{d}{2}-1>p$. By continuation, $\mathcal{H}_{\boldsymbol{\theta}}$ remains p-times differentiable at $h=a$ when either $\beta+\gamma\in\mathbb{N}$ or $\beta+\gamma-\alpha-\frac{d}{2}\in\mathbb{N}$.

Appendix C. Proofs

Lemma 1 (Pólya and Szegö, 1998, p. 81) *Let $\{f_n : n \in \mathbb{N}\}$ be a sequence of real-valued non-increasing functions on $[0, b]$, with $b \in (0, +\infty]$, that converge pointwise to a continuous function f on $[0, b]$. Then, the convergence is uniform on $[0, b]$.* ■

Proof of Theorems 1 and 2 Let $a, \alpha, \beta, \gamma, \tau \in \mathbb{R}_{>0}$, $d \in \mathbb{N}_{\geq 1}$, and $\kappa \geq 0$ such that $2(\beta - \alpha)(\gamma - \alpha) \geq \alpha$, $2(\beta + \gamma) \geq 6\alpha + 1$ and $\alpha - \frac{d}{2} - \kappa \notin \mathbb{N}$. Let $\boldsymbol{\theta} = (a, \alpha, \beta, \gamma, d, \kappa)$ and define the mapping $\widehat{\mathcal{H}}_{\boldsymbol{\theta}}$ through

$$\widehat{\mathcal{H}}_{\boldsymbol{\theta}}(u) = \tau u^{2\kappa} {}_1F_2\left({\alpha \atop \beta,\ \gamma}; -\frac{a^2u^2}{4} \right), \quad u \in (0, +\infty),$$

which is nonnegative on $(0, +\infty)$ owing to Theorem 4.2 in Cho et al. (2020). By expressing the Bessel-J function in terms of the generalized hypergeometric function ${}_0F_1$ (Olver et al., 2010, 10.16.9) and using formulae 5.1 in Miller and Srivastava (1998) (valid under the additional condition $\alpha > \frac{d+1}{4} + \kappa$), the Fourier-Hankel transform (8) of $\widehat{\mathcal{H}}_{\boldsymbol{\theta}}$ at $h > 0$ is found to be

$$\begin{aligned}
\mathcal{H}_{\boldsymbol{\theta}}(h) &= \frac{2\tau\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \int_0^{+\infty} u^{d+2\kappa-1} {}_0F_1\left({- \atop \frac{d}{2}}; -\frac{u^2h^2}{4} \right) {}_1F_2\left({\alpha \atop \beta,\ \gamma}; -\frac{a^2u^2}{4} \right) \mathrm{d}u \\
&= \begin{cases}
\frac{2\tau\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \frac{\Gamma(\frac{d}{2}+\kappa)\Gamma(\frac{d}{2})}{2(h/2)^{d+2\kappa}\Gamma(-\kappa)} {}_3F_2\left({\alpha,\ \frac{d}{2}+\kappa,\ \kappa+1 \atop \beta,\ \gamma}; \frac{a^2}{h^2} \right) \text{ if } a < h \\
\frac{\tau\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \frac{2^{d+2\kappa}\Gamma(\frac{d}{2}+\kappa)\Gamma(\alpha-\frac{d}{2}-\kappa)\Gamma(\beta)\Gamma(\gamma)}{a^{d+2\kappa}\Gamma(\alpha)\Gamma(\beta-\frac{d}{2}-\kappa)\Gamma(\gamma-\frac{d}{2}-\kappa)} {}_3F_2\left({\frac{d}{2}+\kappa,\ 1+\frac{d}{2}+\kappa-\beta,\ 1+\frac{d}{2}+\kappa-\gamma \atop 1+\frac{d}{2}+\kappa-\alpha,\ \frac{d}{2}}; \frac{h^2}{a^2} \right) \\
+\frac{\tau\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \frac{2^{d+2\kappa}\Gamma(\frac{d}{2})\Gamma(\beta)\Gamma(\gamma)\Gamma(\frac{d}{2}+\kappa-\alpha)}{a^{2\alpha}\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)\Gamma(\alpha-\kappa)} h^{2\alpha-d-2\kappa} {}_3F_2\left({\alpha,\ 1+\alpha-\beta,\ 1+\alpha-\gamma \atop 1+\alpha-\frac{d}{2}-\kappa,\ \alpha-\kappa}; \frac{h^2}{a^2} \right) \text{ if } 0 < h < a,
\end{cases}
\end{aligned}$$

which is well-defined for $h \in (0, a) \cup (a, +\infty)$ and extendable at $h = a$ by continuity under the conditions stated above. Now, if $\kappa \in \mathbb{N}$, $\mathcal{H}_{\boldsymbol{\theta}}$ turns out to be identically zero on $(a, +\infty)$, and if $\alpha > \frac{d}{2} + \kappa$, it can be extended by continuity at $h = 0$; in such a case, to obtain $\mathcal{H}_{\boldsymbol{\theta}}(0) = 1$, we have to set

$$\tau = \frac{a^{d+2k}\Gamma(\frac{d}{2})\Gamma(\alpha)\Gamma(\beta - \frac{d}{2} - \kappa)\Gamma(\gamma - \frac{d}{2} - \kappa)}{\pi^{\frac{d}{2}} 2^{d+2\kappa}\Gamma(\frac{d}{2} + \kappa)\Gamma(\alpha - \frac{d}{2} - \kappa)\Gamma(\beta)\Gamma(\gamma)},$$

which yields the announced kernel $\mathcal{H}_{\boldsymbol{\theta}}$ (Theorem 1) and d-radial spectral density $\widehat{\mathcal{H}}_{\boldsymbol{\theta}}$ (Theorem 2). The previous arguments also imply that $\mathcal{H}_{\boldsymbol{\theta}}$ is continuous on $[0, +\infty)$.

Note that conditions (A.1) to (A.4) guarantee the existence of $\mathcal{H}_{\boldsymbol{\theta}}(0)$ and nonnegativity of $\widehat{\mathcal{H}}_{\boldsymbol{\theta}}$, but they may not be necessary. Therefore, some kernels of the $\mathcal{H}$ class may belong to Φ_d without satisfying these conditions. Necessary conditions for $\widehat{\mathcal{H}}_{\boldsymbol{\theta}}$ to be nonnegative are obtained by trading (A.2) for the conditions $\beta > \alpha$ and $\gamma > \alpha$ (Cho et al., 2020).

Proof of Theorem 3 Results in Cho et al. (2020) show that the function $\widehat{\mathcal{H}}_{\boldsymbol{\theta}}$ in (14) is nonnegative if the following conditions hold:

(B.1) $\alpha > 0$;

(B.2) $2(\beta - \alpha)(\gamma - \alpha) \geq \alpha$;

(B.3) $2(\beta + \gamma) \geq 6\alpha + 1$.

Conditions (B.1) to (B.3) are met when conditions (A.1) to (A.3) hold. Additionally, under these conditions, the mapping $\| \cdot \|_d \mapsto \widehat{\mathcal{H}}_{\boldsymbol{\theta}}(\| \cdot \|_d)$ is absolutely integrable in $\mathbb{R}^d$. To prove the claim of the theorem, we invoke the asymptotic expansion of the generalized hypergeometric function ${}_1F_2$ (Mathai, 1993, p. 146):

$$ {}_1F_2\left({\alpha \atop \beta,\ \gamma}; -\frac{x^2}{4} \right) \underset{x \to +\infty}{\sim} A x^{\alpha - \beta - \gamma + \frac{1}{2}} \cos(x + B) + C x^{-2\alpha}, \tag{33} $$

with A, B, C being real values. If condition (B.3) is a strict inequality, then the leading term in (33) is the last term in $x^{-2\alpha}$. Accordingly,

$$ \widehat{\mathcal{H}}_{\boldsymbol{\theta}}(u) \underset{u \to +\infty}{\sim} \zeta_{\boldsymbol{\theta}}\, u^{2k - 2\alpha} $$

with $\zeta_{\boldsymbol{\theta}} \in \mathbb{R}_{>0}$ since $\widehat{\mathcal{H}}_{\boldsymbol{\theta}}$ is strictly positive (Cho et al., 2020, Theorem 4.2). Hence, condition (12) holds for some $0 < c_1 < c_2 < \infty$ and $s = \alpha - k$, with $s > \frac{d}{2}$ under condition (A.1).

This result is no longer guaranteed if condition (B.3) is an equality, in which case

$$ \widehat{\mathcal{H}}_{\boldsymbol{\theta}}(u) \underset{u \to +\infty}{\sim} \zeta_{\boldsymbol{\theta}}\, u^{2k - 2\alpha} (A \cos(au + B) + C) $$

with $\zeta_{\boldsymbol{\theta}} = \widehat{\varpi} a^{d + 2k - 2\alpha} > 0$ and $C \geq |A|$ due to the nonnegativity of $\widehat{\mathcal{H}}_{\boldsymbol{\theta}}$. If $C = |A|$, then condition (12) does not hold. This situation arises, in particular, when $(\beta, \gamma) = (\alpha + \frac{1}{2}, 2\alpha)$ or $(\beta, \gamma) = (2\alpha, \alpha + \frac{1}{2})$, in which case the d-radial spectral density $\widehat{\mathcal{H}}_{\boldsymbol{\theta}}$ has infinitely many zeros on $(0, +\infty)$ (Cho et al., 2020, Remark 4.1).

Proof of Proposition 1 The identity (16) is obtained by use of (29) and formula 15.8.4 in Olver et al. (2010).

Proof of Proposition 2 The identity (17) is obtained by use of (29) and of the series expansion of the hypergeometric function ${}_2F_1$ (Olver et al., 2010, 15.2.1), being terminating series under the conditions stated in the proposition.

Proof of Proposition 3 See Chernih et al. (2014, eq. 15) and Bevilacqua et al. (2024) to establish (18) and the equivalence between conditions (A.1) to (A.3) of Theorem 1 and the stated conditions $\xi > -\frac{1}{2}$ and $\nu \geq \nu_{\min}(\xi, d)$.

Proof of Proposition 4 See Emery et al. (2026) to establish (19) and the equivalence between conditions (A.1) to (A.3) of Theorem 1 and the stated conditions $k \in \mathbb{N}$, $\xi > -\frac{1}{2}$ and $\nu \geq \nu_{\min}(\xi, d + 2k)$.

Proof of Proposition 5 The result stems from (16) and formula 15.4.17 of Olver et al. (2010).

Proof of Proposition 6 This is a particular case of Proposition 4 with $\xi = 0$, see Emery et al. (2026).
Proof of Propositions 7 and 8 See Emery et al. (2026).
Proof of Proposition 9 We just need to establish the convergence of $\mathcal{M}_{a,k+\frac{1}{2},d,k}$ to $\mathcal{J}_{a,d}$: the convergence of $\mathcal{W}_{\mu a,k,\mu,d,k}$ to $\mathcal{J}_{a,d}$ is then a consequence of Proposition 8. The proof relies on the following alternative expression of $\mathcal{M}_{a,\nu,d,k}$ (Emery et al., 2026):

$$\begin{aligned}\mathcal{M}_{a,\nu,d,k}(h) &= {}_1F_2\left(\begin{smallmatrix} k+\frac{d}{2} \\ 1-\nu,\ \frac{d}{2}\end{smallmatrix};\frac{h^2}{4a^2}\right) \\ &\quad + \frac{\Gamma(\nu+\frac{d}{2}+k)\Gamma(\frac{d}{2})\Gamma(-\nu)}{\Gamma(\frac{d}{2}+k)\Gamma(\nu)\Gamma(\nu+\frac{d}{2})}\left(\frac{h}{2a}\right)^{2\nu} {}_1F_2\left(\begin{smallmatrix} \nu+\frac{d}{2}+k \\ \nu+\frac{d}{2},\ \nu+1\end{smallmatrix};\frac{h^2}{4a^2}\right), \quad h \geq 0, \nu \notin \mathbb{N}_{\geq 1}.\end{aligned} \tag{34}$$

Let us write the series representation of the generalized hypergeometric ${}_1F_2$ functions in (34). Concerning the first ${}_1F_2$ function, one has

$$ {}_1F_2\left(\begin{smallmatrix} k+\frac{d}{2} \\ 1-\nu,\ \frac{d}{2}\end{smallmatrix};\frac{h^2}{4a^2}\right) = \sum_{n=0}^{+\infty} \frac{(k+\frac{d}{2})_n}{(1-\nu)_n(\frac{d}{2})_n n!}\left(\frac{h^2}{4a^2}\right)^n, \quad h \geq 0. \tag{35}$$

Let us choose $\nu = k + \frac{1}{2}$. As k tends to infinity, $(k+\frac{d}{2})_n$ and $(1-\nu)_n$ are equivalent to k^n and $(-k)^n$, respectively (Olver et al., 2010, 5.5.3 and 5.11.12). We can therefore split the alternating series (35) into the difference of two series with strictly positive terms, which tend to

$$\sum_{n=0}^{+\infty} \frac{1}{(\frac{d}{2})_{2n}(2n)!}\left(\frac{h^2}{4a^2}\right)^{2n} = \frac{1}{2}\left[{}_0F_1\left(\begin{smallmatrix} - \\ \frac{d}{2}\end{smallmatrix};\frac{h^2}{4a^2}\right) + {}_0F_1\left(\begin{smallmatrix} - \\ \frac{d}{2}\end{smallmatrix};-\frac{h^2}{4a^2}\right)\right]$$

and

$$\sum_{n=0}^{+\infty} \frac{1}{(\frac{d}{2})_{2n+1}(2n+1)!}\left(\frac{h^2}{4a^2}\right)^{2n+1} = \frac{1}{2}\left[{}_0F_1\left(\begin{smallmatrix} - \\ \frac{d}{2}\end{smallmatrix};\frac{h^2}{4a^2}\right) - {}_0F_1\left(\begin{smallmatrix} - \\ \frac{d}{2}\end{smallmatrix};-\frac{h^2}{4a^2}\right)\right],$$

respectively, with all the ${}_0F_1$ terms being non-zero for almost all $h \in [0,+\infty)$. In both cases, the convergence is uniform on $[0,+\infty)$ owing to Lemma 1.

Accordingly, if $\nu = k + \frac{1}{2}$ and k tends to infinity (d, h and a fixed), one obtains (Olver et al., 2010, 10.16.9)

$$ {}_1F_2\left(\begin{smallmatrix} k+\frac{d}{2} \\ \frac{1}{2}-k,\ \frac{d}{2}\end{smallmatrix};\frac{h^2}{4a^2}\right) \to {}_0F_1\left(\begin{smallmatrix} - \\ \frac{d}{2}\end{smallmatrix};-\frac{h^2}{4a^2}\right) = \mathcal{J}_{a,d}(h),$$

the convergence being uniform on $[0,+\infty)$.

Concerning the second hypergeometric function in (34), the series expansion only has positive terms and one obtains, under the same conditions on k and ν:

$$ {}_1F_2\left(\begin{smallmatrix} \frac{d+1}{2}+2k \\ k+\frac{d+1}{2},\ k+\frac{3}{2}\end{smallmatrix};\frac{h^2}{4a^2}\right) \to {}_0F_0\left(\begin{smallmatrix} - \\ -\end{smallmatrix};0\right) = 1,$$

with again the convergence being uniform on $[0, +\infty)$.

The result of the proposition follows from the fact that

$$h \mapsto \left(\frac{h}{2a}\right)^{2k+1} \frac{\Gamma(2k+\frac{d+1}{2})\Gamma(\frac{d}{2})\Gamma(-k-\frac{1}{2})}{\Gamma(k+\frac{d}{2})\Gamma(k+\frac{1}{2})\Gamma(k+\frac{d+1}{2})}$$

uniformly tends to zero on any bounded interval of $[0, +\infty)$ as k tends to $+\infty$.

Proof of Proposition 10 Emery et al. (2026) showed that, for $\nu \geq \frac{3}{2}$, $\mu \geq \max\{\nu + \frac{d}{2}, \frac{1}{a}\}$ and $h \in (0, \mu a - 1)$, one has

$$0 \leq \mathcal{M}_{a,\nu}(h) - \frac{\Gamma(\mu+1)\mu^{2\nu-1}}{\Gamma(\mu+2\nu)} \mathcal{W}_{\mu a, \nu-\frac{1}{2}, \mu, d, 0}(h) \leq \frac{4\nu(\nu+1)}{\mu} + \frac{\Gamma(2\nu-1, \mu)}{\Gamma(2\nu-1)}.$$

Let $a = \frac{b}{2\sqrt{\nu}}$ with $b > 0$ fixed. Also, let ν tend to infinity in such a way that $\mu\nu^{-2}$ tends to infinity. Then,

- $\frac{\Gamma(\mu+1)\mu^{2\nu-1}}{\Gamma(\mu+2\nu)} \to 1$ (Olver et al., 2010, 5.11.12);
- $\frac{4\nu(\nu+1)}{\mu} + \frac{\Gamma(2\nu-1,\mu)}{\Gamma(2\nu-1)} \to 0$;
- $\mathcal{M}_{a,\nu}(h) \to \mathcal{G}_b(h)$ for any $h \in [0, +\infty)$ (Stein, 1999), with the convergence being uniform owing to Lemma 1.

One deduces the uniform convergence of $\mathcal{W}_{\mu a, \nu-\frac{1}{2}, \mu, d, 0}$ to $\mathcal{G}_b$ on $[0, +\infty)$.

Proof of Propositions 11 and 12 Arguments similar to those used in the proof of Proposition 9 allow establishing the following uniform convergences on any bounded interval I of $[0, +\infty)$ as ν tends to infinity:

$${}_1F_2\left(\begin{matrix} \nu+\frac{d}{2}+k \\ \nu+\frac{d}{2},\ \nu+1 \end{matrix}; \frac{\nu h^2}{a^2}\right) \to {}_0F_0\left(\begin{matrix} - \\ - \end{matrix}; \frac{h^2}{a^2}\right) = \exp\left(\frac{h^2}{a^2}\right), \quad h \in I,$$

and

$${}_1F_2\left(\begin{matrix} k+\frac{d}{2} \\ 1-\nu,\ \frac{d}{2} \end{matrix}; \frac{\nu h^2}{a^2}\right) \to {}_1F_1\left(\begin{matrix} k+\frac{d}{2} \\ \frac{d}{2} \end{matrix}; -\frac{h^2}{a^2}\right), \quad h \in I,$$

with the latter expression matching the Gaussian kernel (22) owing to formulae 7.11.1.8 of Prudnikov et al. (1990) and 5.5.3 of Olver et al. (2010). Furthermore, based on formulae 5.5.3 and 5.11.7 of Olver et al. (2010), one has the following uniform convergence on any bounded interval I of $[0, +\infty)$ when ν is a half-integer tending to infinity:

$$\frac{\Gamma(\nu+\frac{d}{2}+k)\Gamma(\frac{d}{2})\Gamma(-\nu)}{\Gamma(k+\frac{d}{2})\Gamma(\nu)\Gamma(\nu+\frac{d}{2})}\left(\frac{\nu h^2}{a^2}\right)^{\nu} \to 0, \quad h \in I.$$

Equation (34) implies the uniform convergence of $\mathcal{M}_{\sqrt{\nu}a,\nu,d,k}$ to $\mathcal{G}_{a,d,k}$ on I as ν is a half-integer tending to infinity. The uniform convergence of $\mathcal{W}_{\mu\sqrt{\nu}a,\nu-\frac{1}{2},\mu,d,k}$ to the same kernel as $\frac{\mu}{\nu}$ also tends to infinity stems from Proposition 8. In passing, these convergences prove that $\mathcal{G}_{a,d,k}$ belongs to Φ_d as a continuous function that is the pointwise limit of members of Φ_d.

Proof of Proposition 13 The proof stems from formulae 8.1.8 of Szegö (1975) and 5.11.12 of Olver et al. (2010).

Proof of Proposition 14 Again, the same argument as in the proof of Proposition 9 allows establishing the following uniform convergences on any bounded interval I of $[0, +\infty)$ as γ tends to infinity and $a/\sqrt{\gamma}$ tends to $b > 0$:

$$ {}_3F_2\left({\alpha,\ 1+\alpha-\beta,\ 1+\alpha-\gamma \atop 1+\alpha-\frac{d}{2}-k,\ \alpha-k} ; \frac{h^2}{a^2} \right) \to {}_2F_2\left({\alpha,\ 1+\alpha-\beta \atop 1+\alpha-\frac{d}{2}-k,\ \alpha-k} ; -\frac{h^2}{b^2} \right), \quad h \in I, $$

$$ {}_3F_2\left({\frac{d}{2}+k,\ 1+\frac{d}{2}+k-\beta,\ 1+\frac{d}{2}+k-\gamma \atop 1+\frac{d}{2}+k-\alpha,\ \frac{d}{2}} ; \frac{h^2}{a^2} \right) \to {}_2F_2\left({\frac{d}{2}+k,\ 1+\frac{d}{2}+k-\beta \atop 1+\frac{d}{2}+k-\alpha,\ \frac{d}{2}} ; -\frac{h^2}{b^2} \right), \quad h \in I, \tag{36} $$

and

$$ \frac{\Gamma(\gamma - \frac{d}{2} - k)}{\Gamma(\gamma - \alpha)} \left(\frac{h}{a}\right)^{2\alpha-d-2k} \to \left(\frac{h}{b}\right)^{2\alpha-d-2k}, \quad h \in I. $$

Accordingly, as γ tends to infinity, the kernel $\mathcal{H}_{\boldsymbol{\theta}}$ as defined in (13) with $a = b\sqrt{\gamma}$ and $\beta = 1 + \frac{d}{2} + k$ uniformly converges on I to the kernel $\mathcal{I}_{b,\alpha,d,k}$ defined by

$$ \begin{aligned} \mathcal{I}_{b,\alpha,d,k}(h) &= 1 - \frac{(\alpha-k)_k}{(\frac{d}{2})_k \Gamma(\alpha - \frac{d}{2} - k + 1)} \left(\frac{h}{b}\right)^{2\alpha-d-2k} {}_2F_2\left({\alpha,\ \alpha-\frac{d}{2}-k \atop 1+\alpha-\frac{d}{2}-k,\ \alpha-k} ; -\frac{h^2}{b^2} \right) \\ &= 1 - \sum_{n=0}^{k} \frac{(-1)^n k! (\alpha-k)_k (\alpha - \frac{d}{2} - k)_n}{n!(k-n)! \Gamma(1 + \alpha - \frac{d}{2} - k + n)\, (\alpha-k)_n (\frac{d}{2})_k} \\ &\quad \times \left(\frac{h}{b}\right)^{2\alpha-d-2k+2n} {}_1F_1\left({\alpha-\frac{d}{2}-k+n \atop 1+\alpha-\frac{d}{2}-k+n} ; -\frac{h^2}{b^2} \right), \end{aligned} $$

where we accounted for the fact that the right-hand side of (36) is identically equal to 1 when $\beta = 1 + \frac{d}{2} + k$, and for Theorem 2.1 of Withers and Nadarajah (2014) to expand the ${}_2F_2$ function into a series of ${}_1F_1$ functions. The claim of the proposition follows from Olver et al. (2010, 8.5.1).

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